ENDOGENOUS DEFAULTS, VALUE-AT-RISK AND THE BUSINESS CYCLE

Issam Samiri
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National Institute of Economic and Social Research
2 Dean Trench St
London SW1P 3HE
T: +44 (0)20 7222 7665
E: enquiries@niesr.ac.uk
www.niesr.ac.uk
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Abstract

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Classification: E13, E32, E44, G21, G32

Keywords: RBC, Value-at-Risk, bank leverage, Credit Spreads, Financial Frictions

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By Issam Samiri†

Draft: April 9, 2024

I propose a general equilibrium model with endogenous defaults and a banking sector operating under a Value-at-Risk constraint. Analytical examination reveals that (a) the Value-at-Risk rule introduces a risk premium on bank lending, (b) this risk premium fluctuates with the business cycle, amplifying the impact of real shocks, and (c) bank leverage also fluctuates with real shocks, but its cyclical behaviour depends on the shocks' effects on default expectations, credit demand, and the bank's balance sheet. Assuming TFP shocks as the sole exogenous source of fluctuation, the model quantitatively replicates realistic fluctuations in banks' leverage, equity, lending, and credit spreads.
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I present a simple general equilibrium model with endogenous defaults in the productive sector and a financial intermediation sector where leverage is constrained by a Value-at-Risk rule. The model is used to study the interaction between the behaviour of financial intermediaries and the business cycle. In the studied framework, unexpected adverse shocks to the real economy push default rates higher. This affects financial intermediaries in two ways. First, higher-than-expected default rates impair bank equity, thus affecting the ability of banks to lend. Under a binding fixed leverage constraint, a relative loss in the bank's equity results in the same relative loss in credit supply. When the bank is subject to a

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† The National Institute of Economic and Social Research, issam.samiri@gmail.com.

Value-at-Risk constraint, higher default expectations can push the bank to reduce leverage, leading to a larger drop in credit supply than what can be explained by the erosion of bank equity. The compounded effects of lower bank equity and lower leverage limits can dry up credit supply, thus pushing financing costs higher. This is the procyclical Value-at-Risk argument already studied in the literature (Adrian and Shin, 2010, 2014; Nuño and Thomas, 2017). In a departure from existing literature, I show that another scenario is also possible. Following large shocks to equity capital, the bank can use leverage to maintain credit supply at a higher level than under a fixed leverage rule. In other words, leverage under a Value-at-Risk rule can move to attenuate shocks (countercyclical leverage).

Financial intermediaries are assumed to fund loan issuance through a combination of deposits and more costly equity financing. The Value-at-Risk constraint on the bank's leverage combined with expensive equity financing introduces a financing cost wedge from the perspective of the borrowing firm that cannot be explained by credit losses alone. This wedge is interpreted as a risk premium cost affecting borrowing in the productive sector. Fluctuations in the banking sector's equity and leverage lead to changes in risk premia. The model, thus, helps to replicate another well-documented feature of credit markets, namely that most of the credit spread fluctuation can be explained by changes to credit risk premia (Gilchrist and Zakrajšek, 2012).

The model's endogenous default mechanism builds on the mechanism in Gourio (2013) by assuming that firms finance capital accumulation using debt financing. Unexpected idiosyncratic shocks to the firm's ability to transform the economy's final good into capital make defaults possible in the steady state. Fluctuations in total factor productivity (TFP) change the value of the capital accumulated in the productive sector, thus causing fluctuations in default rates. A careful specification of the technology used by firms to transform the economy's final good into capital generates rich default dynamics. Unexpected fluctuations in default rates affect the performance of the representative bank's loan portfolio and therefore its equity. On the other hand, the generated default rate distribution also displays some persistence. Persistence of default fluctuations means that the bank's evaluation of future default risk changes over time. The changes to future default expectations interact with the bank's Value-at-Risk constraint generating fluctuations in credit risk premia. The model-implied distribution of default rates is achieved by assuming some curvature in the capital production function and costly adjustment to the firm's investment levels. Both these features are absent in related works (Gourio, 2012, 2013; Nuño and Thomas, 2017). The careful specification of capital production and the resulting rich default rates dynamics enable the model to produce quantitatively reasonable fluctuations of credit spreads, risk premia and leverage from TFP fluctuations as the sole source of exogenous shocks. Unlike other macro-finance models replicating second moments of credit risk premia, the model can therefore do away with additional exogenous sources of fluctuations such as shocks to capital quality, changes to the riskiness of capital production or changes in the likelihood of extreme real aggregate events.

Adrian and Shin (2014) show that, in the short term, banks react to adverse shocks by reducing leverage while keeping equity relatively stable. To reproduce the sticky behaviour of bank equity, I assume that the bank takes one period to adjust its equity capital, by raising new equity or simply through retained earnings. Following shocks that impair the value of the bank's assets, the bank cannot immediately adjust its equity. If it could, it would use the newly raised equity capital to attenuate the effect of lower leverage, thus keeping the change in credit supply minimal. The sticky equity assumption is therefore important to generating realistic fluctuations in leverage and credit spreads.

A generic notion of Value-at-Risk (VaR) is introduced. Under the additional weak assumption that VaR changes linearly with the size of the bank portfolio (linear VaR), The Value-at-Risk constraint can be interpreted as an external upper limit on the bank's financial leverage. Under the linear VaR assumption, I show analytically that the bank lending problem implies a risk premium beyond what can be justified by default risk. The steady-state risk premium results from the combination of expensive equity financing and the limit on bank leverage that makes equity financing necessary to fund new loans. Dynamically, fluctuations in risk premia result from the VaR rule constraining more or less bank lending. A trivial example of a linear VaR measure assumes that VaR is a constant fraction of the portfolio size. Under this rule, bank leverage is constant (acyclical). More generally, the cyclical behaviour of leverage depends on the nature of the Value-at-Risk rule adopted by the bank.

In this paper, I focus on a VaR measure called "economic capital" Value-at-Risk. The economic capital Value-at-Risk (VaR) is aimed at protecting the bank's solvency from a large aggregate shock that affects the value of the bank's assets.

The economic capital VaR constraint requires the bank to hold a large enough equity capital to absorb the drop in value of the bank's assets following a given surprise deterioration in aggregate TFP, thus protecting depositors from making a loss. This requirement is used to derive a formal definition of Value-at-Risk.

I show Analytically that, when constrained by an economic capital VaR, the cyclical behaviour of bank leverage depends on the impact of real shocks on three important variables: the bank's equity, the bank's expectation regarding future defaults and demand for loans. Following an adverse aggregate shock, the bank can increase leverage (countercyclical leverage) if the shock causes a large deterioration in the bank's equity with moderate effects on default expectations and credit demand. On the other hand, if the adverse shock increases future default expectations, or reduces credit demand without much impact on the bank's balance sheet, the bank reacts by reducing leverage (procyclical leverage).

In specifications where leverage is procyclical, the VaR constraint can be interpreted as a procyclical upper limit on the bank's leverage. The procyclical leverage cap compounds the procyclical behaviour of the bank's equity, restricting credit supply in the trough of the business cycle and easing it when the economy is booming. This generates countercyclical risk premia that worsen credit access when default rates are high and vice versa. The Value-at-Risk constraint acts as a financial accelerator, but unlike popular credit accelerator models that focus on the borrower's net worth– e.g., Carlstrom and Fuerst (1997); Kiyotaki and Moore (1997); Bernanke, Gertler and Gilchrist (1998)–the model I present also generates countercyclical credit spreads and credit risk premia.[1]

The model is estimated to quantitatively replicate many salient features of the credit cycles, including realistic dynamics of default rates, credit spreads, bank leverage and lending. Insights gained from the theoretical results are explored further through simulation of the estimated version of the model. Simulations show that the model can produce quantitatively reasonable first and second moments of credit spreads and bank leverage. Moreover, the endogenous bank equity fluctuations and VaR constraint on leverage can significantly affect real variables such as aggregate investment.

In the model I present, lenders operate under a Value-at-Risk constraint limiting their ability to lend. The large TFP shock required for the economic capital Value-

[1] The inability of financial accelerator models to reproduce countercyclical credit spreads is discussed in Gomes, Yaron and Zhang (2003).

at-Risk constraint to generate realistic risk premia is consistent with the economic disasters documented in Barro (2006). Barro (2006) calibrates the average size and occurrence probabilities of economic disasters using data from twentieth-century global history. The author then builds on the previous work of Weil (1989) and Rietz (1988) to design a real business cycle model providing a solution to the equity premium puzzle, i.e., replicating high returns on equity investments, that exceeds by far the low returns on risk-free debt. Following Gourio (2013), I use the likelihood and size of economic disasters documented by Barro (2006) to inform the model's calibration. Gourio (2013) assumes an exogenous time-variable disaster risk driving the bond credit risk premia. In this paper, I show that, when interacting with the lenders' Value-at-Risk rule, the possibility of a disaster-type risk is enough to generate reasonable credit spread and risk premia dynamics, without the need to assume variable disaster probabilities.

The model builds on a rich literature concerned with the role played by the fluctuations in the financial intermediaries' balance sheet in a worsening supply of financing to the economy (e.g., Gertler and Karadi, 2011; Gertler and Kiyotaki, 2010). This paper focuses on the interaction between aggregate real shocks, endogenous defaults and risk-taking rules in the financial intermediation sector. Financial intermediaries can restrict lending when their balance sheet is impaired (contemporaneous change in endogenous default rates) and their appetite for lending is decreased by the interaction of the VaR rule with the endogenous fluctuations of future defaults.

Nuño and Thomas (2017) build on the work of Adrian and Shin (2010, 2014) and present a general equilibrium model with endogenous bank leverage fluctuations. This work departs from the set-up in Nuño and Thomas (2017) in several ways. First, Nuño and Thomas (2017) assume that banks enter into perfect state-contingent debt contracts with the borrowing firms that entitle them to the full cash flow emanating from productive firms in the economy. Their framework assumes no default in the productive sectors. Banks can, however, default as they use debt contracts to partially finance their lending operations. The framework in Nuño and Thomas (2017) cannot be used to model default rates in the corporate sector, nor can it be used to comment on lending risk premia beyond what can be justified by default risk. Moreover, Nuño and Thomas (2017) build on the microfoundations in Adrian and Shin (2014) where the presence of a risk-shifting moral hazard affecting the bank lending decision is used to justify a Value-at-Risk rule. In this paper, all borrowing firms are a priori identical, which excludes the risk-shifting argument as a microfoundation of the Value-at-Risk rule. The bank is, however, exposed to aggregate TFP shocks that affect default rates and thus the performance of its loan portfolio.

Adrian and Duarte (2018) present a continuous-time New Keynesian model featuring financial vulnerability. They introduce a banking sector, tasked with financing the economy's productive sector, that is constrained by an occasionally binding Value-at-Risk rule. They show that monetary policy also reacts to output via the pricing of risk that relates to the tightness of the Value-at-Risk constraints. Unlike in the current work, the authors assume that the Modigliani-Miller theorem applies to both the borrowing firm and the lending bank. They therefore abstract from defaults in the productive sector and from the role played by the bank's capital structure, two aspects at the heart of this work. Moreover, in Adrian and Duarte (2018) fluctuations result from exogenous preference shocks affecting the banks' demand for assets. The model I present here assumes that TFP is the sole source of fluctuations and is thus closer to the existing RBC and DSGE literature.

The remainder of this work is organised as follows. Section II presents the general equilibrium model. Section III is dedicated to presenting several analytical results that help clarify the mechanisms at play. Section IV provides details about the calibration and simulation of the model and comments on the steady-state results and dynamic effects of the main mechanism. Section V concludes. The macro-finance stylised facts are discussed next in the section I.

I. Stylised Facts

This section documents the main stylised facts captured by the model presented here. These relate to the cyclical behaviours of the balance sheets of the financial intermediaries and to the cyclical behaviour of credit spreads.

I consider the balance sheet properties of the aggregate financial intermediation sector composed of financial companies, brokers/dealers and depositary institutions. Table 1 shows the business cycle statistics of the balance sheet of the aggregate financial sector for the period 1981:I–2022:IV. In line with the empirical findings in Nuño and Thomas (2017), the table shows that the assets, equity and leverage of the aggregate financial sector are more volatile than GDP both at the quarterly and annual frequencies.[2] In addition, leverage in the aggregate financial sector is strongly correlated with banks' assets and strongly negatively correlated with equity. Moreover, both assets and leverage are procyclical. Equity, however, does not display a statistically significant correlation with GDP.

TABLE 1—BUSINESS CYCLE STATISTICS, 1981:I-2022:IV

Standard deviation (%)	Quarterly freq.	Annual freq.
Assets	2.7	5.5
Leverage	4.2	6.4
Equity	3.5	4.0
GDP	1.3	1.7
Correlations		
Leverage–Assets	0.58	0.78
	(0.000)	(0.000)
Leverage–Equity	-0.77	-0.52
	(0.000)	(0.000)
GDP–Assets	0.55	0.49
	(0.000)	(0.000)
GDP–Leverage	0.24	0.49
	(0.000)	(0.001)
GDP–Equity	0.11	0.077
	(0.178)	(0.626)

Note: All series are logged and linearly detrended using a Hodrick-Prescott filter ($\lambda = 1600$ for quarterly series and 100 for annual series); p-values of the test of no correlation against the alternative of nonzero correlation are reported in parentheses.
Source: U.S. Flow of Funds and Bureau of Economic Analsyis. See Data Appendix for details.

Figure 1 displays the business cycle components of the aggregate financial sector's leverage, equity and total assets in the period spanning 2006:I to 2012:II. The figure's dynamics are consistent with the business cycle statistics of table 1. All three balance sheet quantities display significant volatility. Leverage appears to be strongly correlated with total assets and negatively correlated with equity. The cyclical components of leverage and total assets appear to peak at the start of the Great Recession. Cohen-Cole et al. (2008) justify this by an expansion of banks' exposure due to firms using pre-existing loan commitments and lines of credit, and securitisation exposure returning to banks' balance sheets. Bank

[2]Unlike Nuño and Thomas (2017) who assume a linear trend of the log variables, detrending here uses a Hodrick-Prescott filter.

equity, on the other hand, experienced a sharp decline at the onset of the Global Financial crisis, then quickly recovered to much higher levels by 2009-II, before the end of the Great Recession. The behaviour of bank equity explains the high volatility of leverage during the Great Recession. Leverage increases more than assets to compensate for lower equity capital, and the cyclical component of leverage turns negative as bank capital recovers. Finally, the acyclical behaviour of the financial sector equity during the Great Recession is consistent with the insignificant equity-GDP correlation figures in table 1; the equity's cyclical component drops as the banks' assets are eroded and recovers as banks raise capital under market pressure (Brei and Gambacorta, 2016).

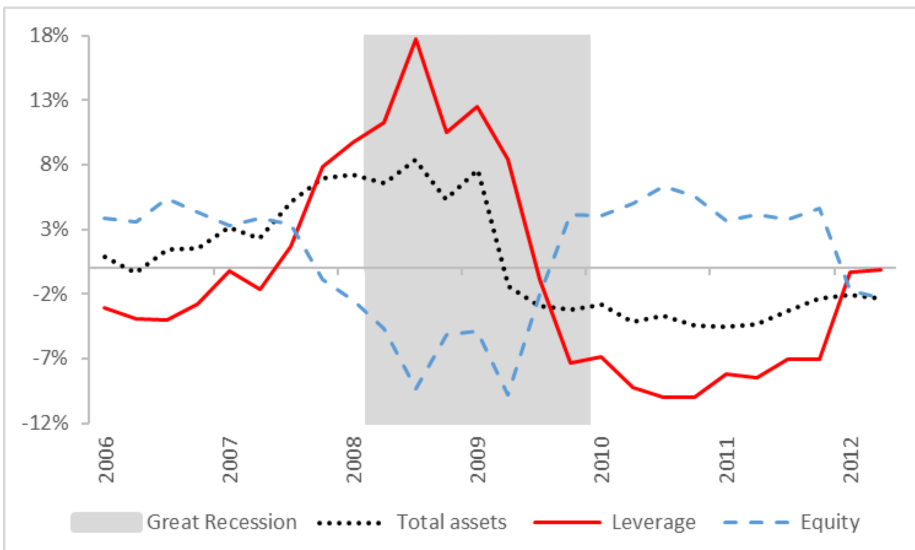

Figure 1. Assets, equity and leverage of the aggregate financial sector.

Note: All variables are expressed in log forms and as a deviation from trends.
Source: U.S. Bureau of Economic Analysis and Moody's.

For the sake of clarity, the main stylised facts regarding the balance sheet of the aggregate financial sector are summarised below.

- Credit Supply is procyclical. Financial intermediaries tend to ration credit supply in periods of weak economic activity and increase credit supply during booms.

- Bank leverage is procyclical. Banks tend to decrease leverage during economic downturns and increase leverage during booms.

- The credit rationing is partly caused by the deterioration of banks' equity capital in the early stages of a recession. Overall, bank equity displays an acyclical behaviour.

The countercyclical behaviour of credit spreads is richly documented in economic literature. Gilchrist and Zakrajšek (2012) decompose credit spreads into a component that can be explained by expected default risk and a residual component, dubbed excess bond premium, that is interpreted as a risk premium attached to lending. Furthermore, they show that most of the volatility of credit spreads finds its origins in the fluctuations of risk premia, as only a small fraction of this volatility can be attributed to changes in default loss expectations. Gourio (2013) notes that given the historical default probabilities of BAA and AAA-rated bonds, default risk justifies about a fifth of the BAA-AAA yield spread. Thus, about four-fifths of the BAA-AAA spread reflects a credit risk premium.

In addition to the behaviour of banks' balance sheets discussed above, the model presented here aims to reproduce several stylised facts related to the cyclical behaviour of credit spreads. Namely, that credit spreads are countercyclical and that their fluctuations are mainly driven by changes to risk premia.

II. Model

I study a general equilibrium model with a productive sector composed of a continuum of firms that produce the economy's final good and transform part of the economy's final output into new capital. Firm-level capital production is funded through non-state-contingent loans issued by a financial intermediation sector. I assume that the financial intermediation sector is composed of a unit-measure continuum of identical banks. The representative bank uses households' deposits and bank equity to fund its lending operations. Some of the borrowing firms can default on their debt obligations when their revenues are too low to cover debt payments, and banks are subject to a Value-at-Risk constraint that limits lending. The economy is subject to aggregate TFP shocks that affect the production of the economy's single consumption good while capital production is subject to firm-level idiosyncratic shocks.

A. Firms

I build on the default mechanism in Gourio (2013) and I assume the existence of a production sector of measure one, populated with firms that transform the economy's final good into capital before producing the final good in the following time period. Banks fully finance the firm's purchase of the goods required for capital production through loan issuance. It takes a firm, indexed i, a single period to make the new capital investment $I_{i,t+1}$, to do so it secures financing through a single-period maturity loan of principal $X_{i,t}$ and uses the following production function to transform the invested quantity of final good $X_{i,t}$ into capital

$$I_{i,t+1} = Z_i X_{i,t}^{\gamma_K} \left\{ 1 - \kappa_K \left(\frac{X_{i,t}}{X} - 1 \right)^2 \right\}, \tag{1}$$

where the index i denotes the firm, Z_i is a firm-specific efficiency factor, κ_K is a parameter reflecting the cost of changing the level of capital production, X is the steady-state loan size, $X_t^{\gamma_K}$ represents the deterministic component of the capital production technology before incurring adjustment costs and $0 < \gamma_K < 1$ a curvature parameter. Furthermore, I assume that capital production efficiencies follow a log-normal process $z_{i,t} := \ln Z_{i,t} = \sigma_K \epsilon_i - \sigma_K^2/2$, where the term ϵ_i reflects the firm's idiosyncratic capital production efficiency. The shocks ϵ_i are normally distributed, independent across firms and independent from other model shocks. The parameter σ_K is a volatility parameter representing the riskiness of capital production. The type of firm ϵ_i is unknown before the loan's maturity. All firms are a priori identical. They all face the same cost of financing R_t and raise the same loan principal X_t.

The firm aggregates the capital it produces with additional capital it rents from households and uses the resulting capital in the production of a final good used by households for consumption and by all firms to make new capital investments. The production of the economy's final good is constrained by a Cobb-Douglas technology

$$Y_{i,t} = Z_t^a K_{i,t}^{\alpha} L_{i,t}^{1-\alpha}, \tag{2}$$

where α is the share of capital, Z_t^a represents aggregate total factor productivity and $K_{i,t}$ is the capital used by the firm. The firm is assumed to combine its own capital investment with the preexisting capital rented from households and use the resulting capital stock to produce. That is

$$K_{i,t} = I_{i,t} + K^R_{i,t}, \tag{3}$$

where $K^R_{i,t}$ denotes the capital rented by firm i from households. The total factor productivity Z^a_t is driven by the AR(1) process $\ln(Z^a_t) = z^a_t = \rho^a z^a_{t-1} + \epsilon^a_t$, where ϵ^a_t are *i.i.d.* shocks with a normal component e^a_t and an independent jump component: $\epsilon^a_t = \sigma_a e^a_t - J_t.dN_t$. The term dN_t represents a Poisson process that takes the value 1 with probability μ_a and value 0 with probability $1 - \mu_a$ and J_t represents the severity of the jump when $dN_t = 1$. The size of the jump J_t is assumed to be independently and identically distributed, $N(\bar{J}, \sigma_J)$.

Production problem

At the time of production, the firm realises its capital production $I_{i,t}$ and sets labour and capital demand, conditional on $I_{i,t}$ to maximise surplus

$$\max_{L_{i,t}, K^R_{i,t}} \pi^I_{i,t}, \tag{4}$$

where $\pi^I_{i,t+1}$ denotes surplus, that is the firm's revenue net of wages and the cost of renting capital

$$\pi^I_{i,t} = Y_{i,t} - W_t L_{i,t} - R^K_t K^R_t. \tag{5}$$

The surplus maximising problem yields

$$W_t = (1-\alpha)\frac{Y_{i,t}}{L_{i,t}} = (1-\alpha)\frac{Y_t}{L_t}, \tag{6}$$

$$R^K_t = \alpha\frac{Y_{i,t}}{K_{i,t}} = \alpha\frac{Y_t}{L_t}. \tag{7}$$

Replace in the surplus expression for the K_t^R using conditon 3 and for W_t and R_t^K using the capital and labour first-order conditions

$$\pi^I_{i,t} = \alpha \frac{Y_t}{K_t} I_{i,t}. \tag{8}$$

The firm makes no profit and no loss from the production of the final good. However, it generates a surplus from the part of capital it owns $I_{i,t}$.

Investment problem

In the investment stage, the firm sets its loan demand to maximise the expected profit, ignoring the possibility of default

$$\max_{X_{i,t}} \mathbf{E}_t \pi_{i,t+1}. \tag{9}$$

The profit $\pi_{i,t+1}$ includes the surplus from producing $\pi^I_{i,t+1}$ and the proceeds from selling the depreciated capital produced using loans raised in the previous period $(1-\delta)I_{i,t+1}$, net of debt payment $R_t X_{i,t}$

$$\pi_{i,t+1} = \pi^I_{i,t+1} + (1-\delta)Q_{t+1} I_{i,t+1} - R_t X_{i,t}, \tag{10}$$

where Q_{t+1} is the price of capital at time $t+1$.[3] Using expression 8, one can write

$$\pi_{i,t+1} = \underbrace{\left\{(1-\delta)Q_{t+1} + \alpha\left(\frac{Y_{t+1}}{K_{t+1}}\right)\right\}}_{:=R^I_{t+1}} I_{i,t+1} - R_t X_{i,t}.$$

For brevity, I note $g(X_t) := X_t^{\gamma_K}\left(1 - \kappa_K(\frac{X_t}{X} - 1)^2\right)$ the deterministic part of the capital production function and $R^I_t := (1-\delta)Q_t + \alpha\left(\frac{Y_t}{K_t}\right)$ the return on investment. This enables us to rewrite the firm's profit function as follows

$$\pi_{i,t+1} = R^I_{t+1} e^{\sigma_K \epsilon_i - \sigma_K^2/2} g(X_{i,t}) - R_t X_t. \tag{11}$$

From 11 and noting the independence of ϵ_i from the remaining shocks affecting the economy, the profit maximisation problem 9 yields a first-order condition

[3] The firm ignores the impact of default on expected profits. See Arellano, Bai and Kehoe (2019) for an alternative assumption where the firm takes the possibility of its own default into account.

common to all firms, confirming that all firms raise the same loan principal X_t

$$R_t = g'(X_t)\mathbf{E}_t R^I_{t+1}. \tag{12}$$

It is worth highlighting that while the firm behaviour adopted here is partly inspired by Gourio (2013), the present specification differs in two important aspects. First, I assume that the firm production of capital only concerns new investments and not the full capital stock. This helps in linking firm borrowing to new investment, thus helping generate quantitatively reasonable dynamics of aggregate lending. In addition, I assume a capital production technology that allows for some curvature in capital production as a function of the loan size X_t ($\gamma_K < 1$). Finally, capital production is subject to adjustment costs when the loan size differs from its steady-state value.[4] As we will see below, these assumptions are key to generating quantitatively reasonable dynamics of default rates and credit spreads.[5]

Endogenous defaults

Default happens when the firm's profit turns negative $\pi_{i,t} < 0$. Upon default, the lending bank takes over the production process and loses a fraction θ_K of the defaulting firm's own capital $I_{i,t}$ in the process, reflecting a cost for the bank to go through bankruptcy workouts and the fact that the firms' managers possess more knowledge about the production process than banks.

The negative profit condition is equivalent to the firm's idiosyncratic shock is lower than the cut-off point $-\xi_t$: $\epsilon_i < -\xi_t$ where

$$\xi_t = \frac{1}{\sigma_K} \ln\left(\frac{R^I_t g(X_{t-1})}{R_{t-1}X_{t-1}}\right) - \frac{\sigma_K}{2}. \tag{13}$$

The firm's default probability is thus given by $\Phi(-\xi_t)$, where Φ denotes the normal cumulative distribution function. Following the nomenclature inspired by Merton (1974), ξ_t is called the "distance to default". In the current set-up, the next period's distance to default depends on the current loan size X_t, the gross

[4]In Gourio (2013), $g(X_t) = X_t$. In line with equation 12, this means that the economy reacts to higher debt costs (higher R_t) by reducing capital investments (higher R^I_{t+1}, thus lower X_t). However, constant returns to scale in capital production imply that the firm size is indeterminate.

[5]Unlike in Gertler and Karadi (2011) the adjustment costs to capital production are not used to impact the price of capital through a net investment first-order condition. In this model, demand for capital emanates from the housing sector (see condition 18 below).

rate of interest R_t and the future return on investment R^I_{t+1}. Note here that $\Phi(-\xi_{t+1})$ is the default probability at the loan's maturity before the firm realizes its type ϵ_i. Once the type ϵ_i is known to the borrowing firm, default or survival are immediately determined and not random any more. Note also that the fluctuation of the return on capital R^I_{t+1} is the only source of changes in the default rates beyond what is expected at the time of loan issuance; all other variables affecting the distance to default ξ_{t+1} are known at the time of issuance. During recessions, the return on investment R^I_{t+1} is low. This reduces the firms' revenues leading to more defaults. Conversely, R^I_{t+1} is higher during booms, leading to fewer defaults in the aggregate. Costly adjustments to the level of capital production have two important implications. First, they dampen fluctuations in capital production. Second, the costly adjustment in capital production pushes firms to maintain a relatively high demand for loans when returns on capital are low. The loan market clears at a higher interest rate R_t, implying higher credit spreads during recessions. The parameter κ_K is key to generating realistic credit spread dynamics.

B. Households

Households like to consume and dislike work as per the utility function

$$U(C_t, L_t) = \frac{C_t^{1-\sigma_H}}{1-\sigma_H} - \chi_H \frac{L_t^{1+\eta_H}}{1+\eta_H}. \tag{14}$$

In addition, households invest in deposits D_t and accumulate capital K_t by buying the capital produced by firms after they exit the market. They decide consumption C_t, labour supply L_t, deposits D_t and new capital purchases I_t by maximising their expected discounted lifetime utility

$$\max_{C_u, L_u, D_u, I_u} \mathbf{E}_t \sum_{u=0}^{\infty} \beta^u U(C_{t+u}, L_{t+u}) \tag{15}$$

where $0 < \beta < 1$ denotes the preferences discount factor. The household's optimisation problem is subject to the budget constraint

$$C_t + D_t + Q_t I_t + E^B_t = w_t L_t + R^D_{t-1} D_{t-1} + R^K_t K_{t-1} + \Pi^B_t + \Pi_t \tag{16}$$

where δ is the depreciation rate of capital, R_t^K is the rental rate of capital, R_t^D is the gross deposit rate, Π_t^B the banks' profit, Π_t the profit distributed by firms and E_t^B the bank's equity. The dynamics of capital are driven by $K_t = (1-\delta)K_{t-1} + I_t$. Finally, the Euler equations for deposits, capital and labour conditions are standard

$$1 = R_t^D \mathbf{E}_t m_{t,t+1}, \tag{17}$$

$$Q_t = \mathbf{E}_t m_{t,t+1}[(1-\delta)Q_{t+1} + R_{t+1}^K], \tag{18}$$

$$\chi_H L_t^{\eta_H} = w_t C_t^{-\sigma_H}, \tag{19}$$

where $m_{t,t+u}$ is the multi-period stochastic discount factor $m_{t,t+u} := \beta^u \frac{C_{t+u}^{-\sigma_H}}{C_t^{-\sigma_H}}$.

C. Banks

The economy under study relies on a representative bank that provides financial intermediation to all firms. The bank issues loans to fund capital production, and its profits are impacted by fluctuations in default rates. To finance loan issuance, the bank raises equity capital and accepts deposits from households. Additionally, bank lending is restricted by a Value-at-Risk constraint.

The bank's profit

Banks hold a balance sheet composed of loans issued to finance capital accumulation and fund these loans using households' deposits. The representative bank invests in a large enough portfolio of loans such as the final fraction defaulting is $\Phi(-\xi_{t+1})$.[6] Upon default, the bank takes over the production process, losing a fraction θ_K of the capital owned by the defaulting firm in the process. We assume, however, that bankruptcy does not impact the final good production beyond its impact on the firm's capital $I_{i,t}$. At the time t, the bank's total recovery proceeds include the surplus from all firms with an idiosyncratic shock ϵ_i smaller than $-\xi_t$, reduced by the fraction θ_K lost after default

[6] This is a direct consequence of the law of large numbers.

$Rec_t := (1-\theta_K)R_t^I g(X_{t-1})\int_{-\infty}^{-\xi_t} Z_{i,t}d\epsilon_i$. Given the distribution of the idiosyncratic shocks ϵ_i and using the definition of the distance-to-default to replace for $Q_t g(X_{t-1})$, the aggregate recovery is[7]

$$(20) \qquad Rec_t = (1-\theta_K)R_{t-1}X_{t-1}e^{\sigma_K\xi_t+\sigma_K^2/2}\Phi(-\xi_t-\sigma_K).$$

The amount recovered by banks increases with the default rate and decreases with the fraction of the defaulting firms lost upon default θ_K.

The bank's profit includes the return from the non-defaulting loans $\Phi(\xi_{t+1})R_tX_t$, the recovery from the defaulting loans Rec_{t+1} and the cost of borrowing from households $R_t^D D_t$

$$(21) \qquad \pi_{t+1}^B = R_tX_t\left\{\Phi(\xi_{t+1}) + (1-\theta_K)e^{\sigma_K\xi_t+\sigma_K^2/2}\Phi(-\xi_t-\sigma_K)\right\} - R_t^D D_t.$$

Bank problem

The bank's lending is subject to the constraint whereby the bank's equity capital must be large enough to absorb potential losses represented by a Value-at-Risk quantity VaR_t to be specified below

$$(22) \qquad VaR_t \leq E_t^B.$$

In addition, the bank's lending operations are also subject to the repeating budget constraints

$$(23) \qquad Div_t + (1+c^B)X_t + R_{t-1}^D D_{t-1} \leq R_{t-1}X_{t-1}\Phi(\xi_t) + Rec_t + D_t,$$

where the right-hand side of the budget constraint 23 reflects the funds available to the bank at time t, including the proceeds from the performing loans issued in the last period $R_{t-1}X_{t-1}\Phi(\xi_t)$, the amount recovered from defaulting loans Rec_t and new deposits D_t. The left-hand side of the budget constraint describes the use of the funds available to the bank at time t. Namely, paying back last period's depositors with interest ($R_{t-1}^D D_{t-1}$), compensating equity capital holders through the dividend distribution Div_t and the funding of new loan and associated lending costs $(1+c^B)X_t$. Furthermore, I assume that the bank can only fund dividends by

[7]See appendix C.C2 for a full derivation.

using a fraction ($d_{t-1} < 1$) of the profits from the last period's lending operation and that the share of profits dedicated to dividends is decided at time of lending

$$Div_t = d_{t-1}\pi_t^B. \tag{24}$$

This latest assumption helps achieve two desirable effects. First, it prevents the use of new deposits to reward existing equity capital. And, crucially, it helps make equity capital sticky, as it prevents the bank from reacting to the deterioration of its equity capital by curtailing dividends or by immediately raising new equity ($Div < 0$). The bank can still improve equity capital by retaining more profits or raising new equity, but it needs to wait until the next period for this to take effect. By definition, the bank's equity is the part of its asset (and lending costs) that is not covered by debt, that is

$$E_t^B = (1 + c_t)X_t - D_t. \tag{25}$$

Plugging the (binding) budget constraint 23 in the definition of the bank equity yields the alternative expression

$$E_t^B = (1 - d_{t-1})\pi_t^B. \tag{26}$$

Expression 26 clarifies that deterioration to the bank's profit due to higher-than-expected default rates will erode the bank's equity capital and that capital is sticky because it will take one period before the bank can raise capital by adjusting d_t. The inability of the bank to immediately replenish its equity capital is in line with the empirical observations in Adrian and Shin (2010, 2014), where the authors show that the first variable of adjustment used by brokers/dealers when adjusting leverage is the assets' side of the balance sheet as opposed to equity capital that remains stable in the short term.

The bank sets the size of its lending operations (X_t) and its future dividends' policy (d_t) by maximising the expected discounted dividends

$$\max_{X_s, d_s} \mathbf{E}_t \sum_{s>t} \frac{m_{t,s}}{(1 + r^E)^{s-t}} d_{s-1}\pi_s^B, \tag{27}$$

subject to the Value-at-Risk constraint 22, the budget constraint 23 and to the dividends constraint 24. Dividends are discounted using the economy's stochastic discount factor $m_{t,s}$ and an extra discount rate $r^E > 0$ reflecting an equity premium over risk-free deposits.[8]

VALUE-AT-RISK

I introduce a generic concept of Value-at-Risk as a measure of the riskiness of the bank's portfolio. A good measure of the risk of the bank's portfolio would depend on the size of the portfolio X_t, the loans' interest rate R_t, and the distribution of distance-to-defaults ξ_{t+1} affecting the performance of the bank's portfolio at time $t+1$.

DEFINITION 1: *A Value-at-Risk measure VaR_t is a non-negative function of the bank's portfolio size X_t, the (gross) loans' interest rate R_t and the distribution of the distance-to-default at time $t+1$, noted $\mathcal{D}(\xi_{t+1})$*

$$VaR_t = VaR(X_t, R_t; \mathcal{D}(\xi_{t+1})). \tag{28}$$

Furthermore, I assume that VaR is $\mathcal{C}^1$ in X_t and R_t, increasing in X_t and decreasing in R_t. A unit Value-at-Risk measure is defined as

$$VaR_t^U = VaR_t / X_t, \tag{29}$$

where VaR_t is a Value-at-Risk measure as defined above.

Definition 1 allows for the VaR measure to be non-positive, in that case, the bank's lending is unconstrained by the Value-at-Risk rule. It is reasonable to assume that the measure of portfolio risk increases with the portfolio size ($\frac{\partial VaR}{\partial X_t} \geq 0$). Similarly, all other things being equal, a portfolio providing higher income before defaults is preferable from a VaR perspective ($\frac{\partial VaR}{\partial R_t} \leq 0$). An additional reasonable feature of VaR_t measures could be that they increase when the distribution $\mathcal{D}(\xi_{t+1})$ implies that high default scenarios are more likely at time $t+1$. I abstract from making this assumption as it is not needed for the model's derivations. In the remainder of this section, I introduce two examples of VaR measures verifying the conditions in definition 1.

[8]This financing friction can be justified by the asymmetry of information issues as discussed in Holmström and Tirole (1998).

Constant leverage Value-at-Risk

A simple VaR rule can emanate from a constant leverage constraint, defined as the ratio of the bank's exposure to its equity capital

$$\lambda_t^B \leq \lambda^{Cst}. \tag{30}$$

To embed this, rather simple, leverage constraint in the current set-up where we express limits on lending in terms of Value-at-Risk, we can rewrite it as follows

$$VaR^{Cst} \leq E_t^B, \tag{31}$$

where by definition $VaR^{Cst} := X_t/\lambda^{Cst}$. A constant constraint on leverage can be therefore seen as equivalent to a Value-at-Risk rule, where the Value-at-Risk is a constant share of the bank's exposure. VaR^{Cst} is linear in the size of the portfolio X_t, with linearity coefficient being the corresponding unit Value-at-Risk is $VaR^{Cst,U} := 1/\lambda^{Cst}$. Clearly, VaR^{Cst} verifies the conditions in definition 1. While this measure does not depend on the loan rate R_t, it is still (weakly) decreasing in R_t. Finally, VaR^{Cst} does not depend on the distribution $\mathcal{D}(\xi_{t+1})$; it is a risk-insensitive VaR measure.

Economic capital Value-at-Risk

I now specify the notion of economic capital Value-at-Risk. This definition is derived from the notion of economic capital as a way to guarantee that the financial intermediary can withstand a certain adverse scenario and continue operating as a going concern. Formally, the financial intermediary's capital should be large enough so that when the assets depreciate under an adverse risk scenario, their value still suffices to cover the financial intermediary's debt obligations, where the adverse risk scenario is defined as the $1 - p^{VaR}$ percentile of the period's $t+1$ distribution of the bank assets' value

$$\underbrace{R_t X_t \Phi(\xi_t^{VaR}) + Rec_t^{VaR}}_{t+1 \text{ assets at VaR scenario}} \geq \underbrace{R_t^D D_t}_{t+1 \text{ liabilities}} . \tag{32}$$

The superscript $.^{VaR}$ is used to indicate the value of the relevant model variables under the Value-at-Risk scenario. ξ_t^{VaR} and Rec_t^{VaR} refer to $t+1$ distance to default and recovery at the VaR scenario respectively

$$\xi_t^{VaR} := \frac{1}{\sigma_K} \ln \left(\frac{R_t^{I,VaR} g(X_t)}{R_t X_t} \right) - \frac{\sigma_K}{2}, \tag{33}$$

$$Rec_t^{VaR} := (1-\theta_K) R_t^{I,VaR} g(X_t) \int_{-\infty}^{-\xi_t^{VaR}} e^{\sigma_K \epsilon_i} d\epsilon_i, \tag{34}$$

where $R_t^{I,VaR}$ is the return on capital investment at VaR scenario

$$R_t^{I,VaR} = R_{t+1}^I (\epsilon_{t+1}^a = -\epsilon^{a,VaR}), \tag{35}$$

$-\epsilon^{a,VaR}$ the next period's innovation to log-TFP corresponding to the $1-p^{VaR}$ quantile $\epsilon^{a,VaR} := -F^{-1}(1-p^{VaR})$ and F is the cumulative distribution of TFP innovations. Condition 32 is equivalent to the bank's asset being larger than its liabilities in the next period with probability p^{VaR}. In other words, condition 32 restricts the bank's lending to maintain a bank insolvency probability in the next period that is less than $1-p^{VaR}$. For example, when the bank is contained by a $p^{VaR} = 95\%$ VaR, the bank's own default probability is limited to 5%. Note how Value-at-Risk variables referring to default risk at time $t+1$ take the time index t. This notation is adopted to indicate that these quantities are known at time t and that they are crucial for credit issuance at time t. Replacing for D_t in 32 using the balance sheet equation $(1+c^B)X_t = D_t + E_t^B$ and rearranging

$$\underbrace{(1+c^B)X_t - \frac{R_t}{R_t^D} X_t \Phi(\xi_t^{VaR}) - \frac{1}{R_t^D} Rec_t^{VaR}}_{:=VaR_t^{EC}} \leq E_t^B. \tag{36}$$

Using the recovery expression 20 leads to the definition below of the economic capital Value-at-Risk as the shortfall in the discounted next period's value of the bank assets below its portfolio notional including lending costs $(1+c^B)X_t$ at the VaR scenario.[9]

$$VaR_t^{EC} := X_t \left\{ 1 + c^B - e^{S_t - S_t^{VaR}} \right\}, \tag{37}$$

[9]See appendix C.C3 for full derivation.

where S_t^{VaR} is a credit spread representing the riskiness of the VaR scenario

$$S_t^{VaR} := -\ln\left(\Phi(\xi_t^{VaR}) + (1-\theta_K)\mathbf{e}^{\sigma_K \xi_t^{VaR} + \sigma_K^2/2}\Phi(-\xi_t^{VaR} - \sigma_K)\right), \tag{38}$$

ξ_t^{VaR} is the distance to default assuming that log-TFP drops by ϵ^{VaR} at time $t+1$ and S_t is the loan's credit spread $S_t := \ln\left(\frac{R_t}{R_t^D}\right)$. Furthermore, the economic capital Value-at-Risk per dollar lent (or unit VaR) $VaR_t^{EC,U} := VaR_t^{EC}/X_t$ is given by

$$VaR_t^{EC,U} := 1 + c^B - e^{S_t - S_t^{VaR}}. \tag{39}$$

Note that the definition of the unit Value-at-Risk $VaR_t^{EC,U}$ enables the rewriting of the VaR constraint as an upper limit on the bank's leverage $\lambda_t^B := X_t/E_t^B$

$$\lambda_t^B \le \left\{1 + c^B - e^{S_t - S_t^{VaR}}\right\}^{-1}. \tag{40}$$

The inequality 40 provides a relationship between three important quantities: the credit spread S_t, the bank leverage λ_t^B and the riskiness of the VaR scenario, represented by S_t^{VaR}. When the VaR constraint is binding, a riskier Value-at-Risk scenario (higher S_t^{VaR}) reduces leverage. On the other hand, higher credit spreads S_t, increase leverage at the constraint. All other things being equal, higher net income from lending increases expected profits, thus providing the bank with a cushion enabling it to absorb more losses without impairing deposits. This effect is important from a macroprudential perspective. In the lows of the cycle, the Value-at-Risk is higher, leading to lower leverage. Lower leverage depresses the supply of credit, thus raising credit spreads higher. However, higher credit spreads help relieve the pressure on leverage, thus increasing credit supply and reducing credit spreads. The effect of higher credit spreads in relaxing the tight leverage constraint can help the economy escape the leverage and risk premia doom loop described in Geanakoplos (2010). The VaR function defined in equation 37 is somewhat similar to the one introduced in Adrian and Duarte (2018), as in both definitions VaR/X_t decreases with the expected returns on the bank's portfolio and increases with the severity of the VaR scenario.[10]

[10]The VaR notion introduced in 37 relates to the asset's holding period, while Adrian and Duarte (2018) refer to a typically shorter risk horizon.

D. Aggregation and Market Clearing

In this subsection, I clarify the market clearing conditions. These are

- The clearing of the labour market.
- The clearing of the loan market where the supply of loans by banks meets the demand for credit.
- The clearing of the market for physical capital where the households' demand for newly created capital meets the new capital produced by all firms and sold after the end of the final production process

$$K_t - (1-\delta)K_{t-1} = \int_{\epsilon_i} I_{i,t} dF(\epsilon_i) - \theta_K \int_{\epsilon_i < -\xi_t} I_{i,t} dF(\epsilon_i).$$

- The clearing of the final goods market $Y_t = C_t + (1 + c^B)X_t$, where $Y_t = \int_{\epsilon_i} Y_{i,t} dF(\epsilon_i) = Z_t^a K_t^\alpha L_t^{1-\alpha}$ is the aggregate output.

III. Analytical results

This section provides several theoretical results clarifying the two main mechanisms at play in the model. First I provide results describing the implications of the chosen endogenous defaults mechanism. Then, I turn to the Value-at-Risk constraint and its effects on leverage and risk premia.

A. Defaults rates dynamics and steady-state results

The proposition below provides an expression of the steady-state distance to default ξ and describes its dynamics outside the steady state.

PROPOSITION 1: *The steady-state distance to default is determined by the parameter driving the curvature of the capital production function γ_K and the parameter driving the idiosyncratic risk associated with capital production σ_K*

$$\xi = -\frac{\ln(\gamma_K)}{\sigma_K} - \frac{\sigma_K}{2}. \tag{41}$$

Dynamically, default rates differ from their steady-state level due to unexpected changes to the return on capital investment R^I_{t+1} or as a result of costly adjust-

ments in capital production

$$(42)\qquad \xi_{t+1} - \xi = \frac{1}{\sigma_K}\ln\left(\frac{R^I_{t+1}}{\mathbf{E}_t R^I_{t+1}}\right) - \frac{1}{\sigma_K}\ln\left(1 - \frac{2\kappa_K}{\gamma_K}\frac{(X_t/X)\{X_t/X-1\}}{1+\kappa_K\{X_t/X-1\}^2}\right).$$

For small movements, the distance-to-default deviation from the steady state can be approximated as below

$$(43)\qquad \xi_{t+1} - \xi \approx \underbrace{\frac{1}{\sigma_K}\left\{\ln(R^I_{t+1}) - \ln(E_t[R^I_{t+1}])\right\}}_{\textit{unexpected shocks' effect}} + \overbrace{\frac{2\kappa_K}{\sigma_K\gamma_K}\left\{\ln(X_t) - \ln(X)\right\}}^{\textit{effect of costly adjustments}}.$$

PROOF:

See appendix C.C4.

Unsurprisingly, proposition 1 implies that steady-state default probability increases with the riskiness of the borrowing firm's capital production process represented by the parameter σ_K. In addition, the steady-state default rate increases with the capital production function curvature parameter γ_K. This behaviour results from the impact of γ_K on the firm's expected profit. Assume that the economy is at the steady state at time t, where $\mathbf{E}_t R^I_{t+1} = R^I$. At the steady state, the firm's investment first-order condition 12, implies that the firm's expected profit is $\mathbf{E}_t\pi_{t+1} = (1-\gamma_K)X^{\gamma_K}R^I$. When capital production is close to displaying constant returns to scale ($\gamma_K \approx 1$), the borrowers' expected profit approaches zero. This decreases the borrowing firm's cushion against unexpected future drops in demand, making defaults more likely. Conversely, lower γ_K implies higher expected profits, thus making negative profits less likely, which means lower default rates.

In a dynamic setting, Proposition 1 decomposes the fluctuations in default rates into two distinct components. The first component is a surprise shock to the economy that creates a wedge between the return on capital and prior expectations. The second component arises from costs affecting the ability of borrowing firms to freely adjust capital production levels. When the economy is in a steady state at time t, any deviation of the distance to default from its steady-state value is solely driven by unexpected changes to the return on capital $\xi_t - \xi = \frac{1}{\sigma_K}\left\{\ln(R^I_t) - \ln(R^I)\right\}$. While lower values of σ_K decrease the steady-state default, dynamically, they increase the volatility of default rates given the dynamics of the return on capital investments R_t^I. The volatility parameter σ_K is therefore key to the volatility of default rates.

Assuming no further unexpected shocks after time $t+1$, the fluctuations of default rates originate from capital production adjustment costs: $\xi_{t+1} - \xi \approx \frac{2\kappa_K}{\sigma_K \gamma_K} \{\ln(X_t) - \ln(X)\}$. After an unexpected shock, default rates are higher if the unexpected shock causes lower capital production at time t. For reasonable calibrations, this would be the case after an unexpected adverse shock to aggregate productivity Z_t^a. The adjustment cost κ_K is, therefore, useful in calibrating the persistence of default rate fluctuations. The credit risk of loan contracts issued at time t is driven by default rates at time $t+1$. The parameter κ_K is key for the dynamics of the fundamental parts of credit spreads.[11]

Solution of the bank problem

As discussed above, the Value-at-Risk constraint can be interpreted as a constraint on the bank leverage $\lambda_t^B := X_t / E_t^B$

$$\lambda_t^B \leq 1/VaR_t^U, \tag{44}$$

where $VaR_t^U := VaR_t / X_t$ is the unit Value-at-Risk. The constraint 44 is easier to interpret as an upper limit on leverage when VaR_t^U is independent of the bank's equity and lending decision. This condition is formalised in the definition below.

DEFINITION 2: *A Value-at-Risk measure VaR_t is linear when the corresponding unit Value-at-Risk $VaR_t^U := VaR_t / X_t$ is independent of the size of the bank's portfolio, i.e.*

$$\frac{\partial VaR_t^U}{\partial X_t} = 0. \tag{45}$$

When the Value-at-Risk measure is linear, the constraint it represents on lending is equivalent to an external time-dependent upper limit on bank leverage

$$\lambda_t^B \leq \frac{1}{VaR^U(R_t, \mathcal{D}(\xi_{t+1}))}, \tag{46}$$

[11] As we will see below, κ_K is also important to the fluctuations of risk premia.

where by definition $VaR^U(R_t, \mathcal{D}(\xi_{t+1})) := VaR(X_t, R_t, \mathcal{D}(\xi_{t+1}))/X_t$. Note that the Value-at-Risk measure VaR^{Cst} introduced in section II.C is linear in X_t. To simplify the exposition of the model results, I restrict attention to this class of VaR measures.

ASSUMPTION 1: *The representative bank's lending is subject to a linear Value-at-Risk constraint.*

Furthermore, I assume that the bank neglects the effect of incremental lending on expected default rates. This assumption helps correct for an important shortfall of this model whereby firms do not accumulate borrowed capital during their lifetime. The firm borrows once, produces capital then produces the economy's final good before winding down its activity. This means that the effect of new lending on the firm defaults is exaggerated in the current framework. Moreover, this assumption greatly simplifies the model's exposition and the derivation of analytical results.

ASSUMPTION 2: *The representative bank neglects the effect of its lending on expected default rates.*

Under Assumption 2, the economic capital VaR introduced in II.C is also a linear VaR measure.[12] Under both assumptions 1 and 2, the leverage constraint is external to the bank's problem, and the bank can neglect the effects of its lending on default expectations. This simplifies the bank's problem, as shown in the proposition below. The proposition states that the bank sets the next period's dividend ratio such as the next period's discounted equity is equal, in expectation, to the next period's discounted bank's value.

PROPOSITION 2: *Assuming that the VaR constraint binds near the steady state, then near the steady state, the bank sets the dividend ratio d_t such as*

$$\mathbf{E}_t m_{t,t+1} E^B_{t+1} = \mathbf{E}_t m_{t,t+1} V^B_{t+1}, \tag{47}$$

where V^B_t designates the bank's value

$$V^B_t = \mathbf{E}_t \frac{m_{t,t+1}}{1 + r^E}(d_t \pi^B_{t+1} + V^B_{t+1}). \tag{48}$$

[12]Note that the economic capital unit Value-at-Risk $VaR^{EC,U}$ is fully determined by credit spreads and expectations over the next period's default rates (equation 39).

PROOF:

See appendix C.C5.

To isolate the effects of the Value-at-Risk constraint, I consider the bank's problem with unconstrained leverage. The bank would still maximise the expected discounted dividends as per the optimisation problem in expression 27, where the optimisation is subject to the budget constraint alone. The first-order conditions yield that the bank dedicates the full bank's profit to pay dividends ($d_t = 1$), implying no bank equity and full financing of loan issuance using deposits. Lending is determined by the first-order condition[13]

$$R_t = (1 + c^B) R_t^{Def} = (1 + c^B) R_t^D e^{S_t^{Def}}, \tag{49}$$

where "the fundamental loan interest rate" R_t^{Def} is the interest rate that implies zero expected profits on the bank's lending operations, ignoring lending costs[14]

$$R_t^{Def} := \frac{R_t^D}{\mathbf{E}_t \left[\Phi(\xi_{t+1}) + (1 - \theta_K) e^{\sigma_K \xi_{t+1} - \sigma_K^2/2} \Phi(-\xi_{t+1} - \sigma_K) \right]} \tag{50}$$

and S_t^{Def} is the corresponding credit spread can be defined as $S_t^{Def} := \ln \left(\frac{R_t^{Def}}{R_t^D} \right)$. In other words, this first-order condition 49 sets the loan's interest rate R_t such as lending revenues exactly compensate for expected default-related losses. In the studied framework, credit risk premia beyond the costs of lending c^B are a consequence of the VaR constraint.

PROPOSITION 3: *The following results stand in the steady state*

- *The fraction of the bank's profit dedicated to dividends d_t is $d = 1 - \frac{\beta}{1+r^E}$.*

- *The credit spread charged by the bank is S given by*

$$S = S^{Def} + \ln \left[1 + c^B + r^E . VaR^U \right], \tag{51}$$

[13] The derivation of the first-order conditions when the bank is not subject to a VaR constraint is in appendix C.C6

[14] Definition 50 can be rewritten as $\mathbf{E}_t \left[R_t^{Def} X_t \Phi(\xi_{t+1}) + Rec_{t+1} - R_t^D X_t \right] = 0$.

where S^{Def} is the credit spread implying zero expected losses from lending

$$S^{Def} = -\ln\left(\Phi(\xi) + (1-\theta_K)e^{\sigma_K\xi+\sigma_K^2/2}\Phi(-\xi-\sigma_K)\right). \tag{52}$$

The risk premium is thus

$$S^{Prem} := S - S^{Def} = \ln\left[1 + c^B + r^E.VaR^U\right]. \tag{53}$$

PROOF:

See appendix C.C7.

Proposition 7 provides two important results. First, the fraction of bank profits dedicated to reward shareholders through dividend distribution decreases with the patience of the households (higher β) and increases with bank equity premium r^E. Second, the credit spread premium S^{Prem} charged by banks reflects the cost of lending c^B and the interaction of expensive equity financing ($r^E > 0$) and the riskiness of the Value-at-Risk scenario represented by VaR^U. The restriction on bank leverage only matters if equity financing is more expensive than financing loans using deposits ($r^E > 0$). When equity financing costs the same as the interest rate paid to depositors ($r^E = 0$), the steady-state credit spread premium solely reflects the cost of lending c^B. Similarly, the credit spread premium solely reflects the cost of lending when leverage is very high ($VaR^U \approx 0$).

When the bank is subject to a binding linear Value-at-Risk constraint, its supply of credit is determined by its level of equity and the leverage upper limit

$$X_t = E_t^B/VaR_t^U. \tag{54}$$

Given the sticky equity assumption, the level of equity is determined by the previous dividend ratio d_{t-1} and the performance of the bank's existing lending portfolio π_t^B. At the VaR constraint, credit supply is fully determined by the previously determined dividend ratio, the performance of the previously constituted bank portfolio π_t^B and the potentially risk-sensitive VaR

$$X_t = \frac{(1-d_{t-1})\pi_t^B}{VaR_t^U}. \tag{55}$$

When the VaR constraint is not binding, the loan interest rate must satisfy $R_t = (1+c^B)R_t^{Def}$. When $R_t < (1+c^B)R_t^{Def}$, lending is not profitable in expectation and the bank refrains from lending. Finally, if and when, for large enough values of R_t, VaR_t^U becomes nonpositive, bank lending is unconstrained and the bank would meet any level of credit demand. Proposition 4 provides a full characterisation of the bank's credit supply function when subject to a linear Value-at-Risk rule.

PROPOSITION 4: *The bank's loan supply function can be described as follows*

(i) The bank does not supply credit when $S_t < S_t^{Def} + \ln(1 + c^B)$.

(ii) The bank supply meets any level of loan demand up to $\frac{(1-d_{t-1})\pi_t^B}{VaR^U((1+c^B)R_t^{Def}, \mathcal{D}(\xi_{t+1}))}$, *charging the loan rate* $R_t^{Def}(1+c^B)$, *which is equivalent to charging the credit spread* $S_t = S_t^{Def} + \ln(1 + c^B)$.

(iii) The bank supply is constrained by the Value-at-Risk constraint when the interest rate is such as the credit spread verifies $S_t > S_t^{Def} + \ln(1 + c^B)$. *In this case, the bank supply of loan is given by the unique solution of the equation*

$$X_t = (1 - d_{t-1})\pi_t^B / VaR_t^U. \tag{56}$$

(iv) For interest rates R_t *such as* $VaR_t^U \leq 0$, *the bank supplies any level of credit demand as long as the loan interest rate implies non-negative profit* $R_t \geq R_t^{Def}(1 + c^B)$.

PROOF:

See appendix C.C8.

In the proposition above, (i) corresponds to the case where the representative bank does not supply credit because the loan rate is not enough to compensate for expected default losses. Case (ii) is where the bank's credit supply is not constrained by the VaR limit. When bank lending is limited by the VaR constraint, credit supply is determined by the VaR constraint and credit spreads exceed the level required to compensate for future default losses (case (iii)). Finally, when the loan rate R_t is high enough and the unit VaR turns non-positive, the bank can meet any level of credit demand at an interest rate implying a non-negative profit (case (iv)).

B. Economic capital Value-at-Risk

I now turn to the economic capital VaR introduced in section II.C and start by making the assumption below regarding the return on capital at the VaR scenario. The assumption is equivalent to a log-linear, first-order approximation of the return on capital R_t^I near the steady state.

ASSUMPTION 3: *Assume that the economy's agents estimate the Value-at-Risk return on capital investment scenario through the log-linear relationship*

$$\ln(R_t^{I,VaR}) = \ln(\mathbf{E}_t[R_{t+1}^I]) - \alpha_{R^I}\epsilon^{VaR}, \tag{57}$$

where α_{R^I} is a constant.

Assumption 3 enables us to write the economic capital VaR distance-to-default as a function of the next period defaults expectations and the model's parameters.

PROPOSITION 5: *Under assumption 3, the VaR default rate scenario fluctuations are solely driven by changes in the next period distribution of defaults*

$$\xi_t^{VaR} = \frac{1}{\sigma_K}\ln\left(\mathbf{E}_t e^{\sigma_K \xi_{t+1}}\right) - \Delta\xi^{VaR}, \tag{58}$$

where $\Delta\xi^{VaR} = \frac{\alpha_{R^I}}{\sigma_K}\epsilon^{VaR}$ is a constant that depends on the severity of the Value-at-Risk adverse shock scenario ϵ^{VaR}.

PROOF:

See appendix C.C9.

A first-order approximation of equation 58 clarifies the result of proposition 5

$$\xi_t^{VaR} \approx E_t[\xi_{t+1}] - \Delta\xi^{VaR}. \tag{59}$$

The distance-to-default at the Value-at-Risk scenario is reduced below its expected level by a quantity that depends on the drop in log TFP (ϵ^{VaR}) associated with the Value-at-Risk scenario. To bring the behaviour of this model closer to that in Gourio (2013), where changes in credit premia are modelled as originating from fluctuations in the likelihood of large adverse shocks to the economy, we could assume that the probability of the jump affecting TFP μ_t is stochastic and driven by an exogenous process. Alternatively, we could assume that the calculation of the VaR is based on a scenario with fluctuating severity to reflect changes in the risk aversion of financial intermediaries. This could be achieved by assuming that the VaR probability (p^{VaR}) is driven by an exogenous process that causes financial intermediaries to index their risk scenario to different quantiles of the TFP distribution.[15]

The proposition below rewrites the results of proposition 4 when the bank operates under an economic capital VaR. It expresses credit supply as a function of the loan credit spread.[16]

PROPOSITION 6: *The bank's loan supply function can be described as follows*

- *The bank does not supply credit when* $S_t < S_t^{Def} + \ln(1+c^B)$.
- *The bank supply meets any level of loan demand up to* $\frac{E_t^B/(1+c^B)}{1-\exp(S_t^{Def}-S_t^{VaR})}$, *charging the credit spread* $S_t = S_t^{Def} + \ln(1+c^B)$.
- *The bank supply is constrained by the Value-at-Risk constraint when the interest rate is such as* $S_t^{Def} + \ln(1+c^B) < S_t < S_t^{VaR} + \ln(1+c^B)$. *In this case, the bank supply of loan is given by*

$$X_t = \frac{E_t^B}{1 + c_t^B - \exp(S_t - S_t^{VaR})}. \tag{60}$$

- *The bank's supply of loans becomes infinitely high as the charged credit spread* S_t *approaches the limit* $S_t^{VaR} + \ln(1+c^B)$ *from below; the credit spread never reaches or surpasses the limit value* $S_t^{VaR} + \ln(1+c^B)$.

PROOF:

Replace for VaR^U in proposition 4 using equation 39.

Figure 2 shows that loan supply by the representative bank is increasing in the credit spread S_t and implies an acceptable range for credit spreads between $S_t^{Def} + \ln(1+c^B)$ and $S_t^{VaR} + \ln(1+c^B)$. Unsurprisingly, the bank refrains from supplying credit when the price of credit is not enough to compensate for lending costs and for expected default losses ($S_t < S_t^{Def} + \ln(1+c^B)$). The bank

[15] To explore the nonlinear effects, write the second-order approximation of equation 58 $\xi_t^{VaR} \approx E_t[\xi_{t+1}] - \Delta\xi^{VaR} + \frac{\sigma_K}{2}\left\{E_t[\xi_{t+1}^2] - E_t[\xi_{t+1}]^2\right\}$. The magnitude of the second-order effects depends on σ_K and the variance of ξ_{t+1}.

[16] This is equivalent to supplying credit at a given loan rate level. There is a one-to-one relationship between the credit spread S_t and the loan rate $R_t = R_t^D e^{S_t}$.

meets any level of credit demand without charging a VaR-related risk premium ($S_t = S_t^{Def} + \ln(1 + c^B)$) up to $\frac{E_t^B/(1+c^B)}{1-\exp(S_t^{Def} - S_t^{VaR})}$ before its VaR constraint binds. Credit supply is then constrained by the bank's VaR rule and is increasing in the credit spread S_t. When the VaR constraint is binding, the bank charges a credit spread premium $S_t^{Prem} := S_t - S_t^{Def}$ that exceeds what is required to cover lending costs $S_t^{Prem} > \ln(1 + c^B)$. Finally, the bank will meet any credit demand for a credit spread lower than $S_t^{VaR} + \ln(1 + c^B)$ and the market's credit spread never reaches the latter limit.

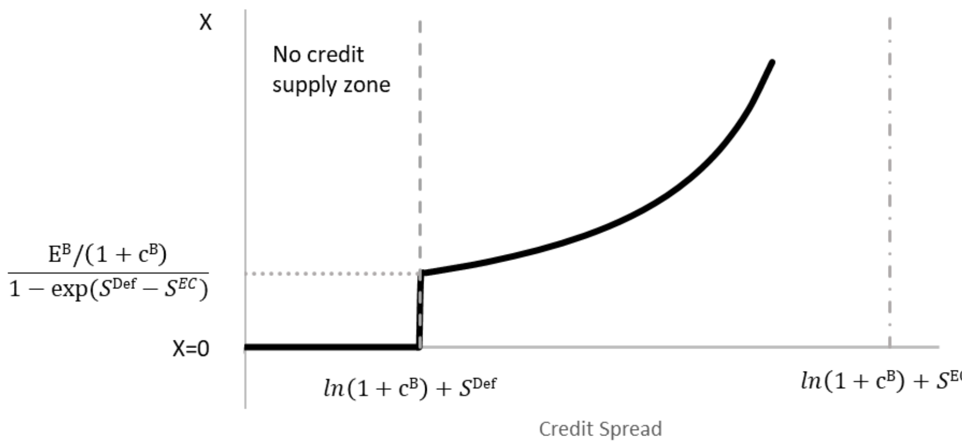

Figure 2. Credit supply by the representative bank

Proposition 7 expresses the steady-state interest rate charged to the borrowing firm, corrected for lending costs, as a weighted harmonic mean of the fundamental interest rate required to compensate for future defaults R^{Def} and the Value-at-Risk scenario interest rate $R^{VaR} = R^D e^{S^{VaR}}$.

PROPOSITION 7: *Assuming the VaR constraint is binding in the steady state, the steady-state loan interest the bank is R given by*

$$\frac{1}{R} = \frac{1}{1 + c^B} \left\{ \frac{1}{1 + r^E} \frac{1}{R^{Def}} + \frac{r^E}{1 + r^E} \frac{1}{R^{VaR}} \right\}, \tag{61}$$

where R^{Def} is the steady state-value of R_t^{Def} defined in 50, R^{VaR} is the steady-state value of $R_t^{VaR} := R_t^D e^{S_t^{VaR}}$ and S_t^{VaR} is defined by expression 38. In credit *spread terms, one can write*

$$e^{-S} = \frac{1}{1+c^B}\left\{\frac{1}{1+r^E}e^{-S^{Def}} + \frac{r^E}{1+r^E}e^{-S^{VaR}}\right\}, \tag{62}$$

where $S^{Def} := \ln(R^{Def}/R^D)$.

PROOF:

Replace for VaR^U in proposition 3 using equation 39.

Note that the VaR scenario impacts the steady-state loan rate more when the bank's equity capital is expensive (high r^E). In the absence of an equity risk premium ($r^E = 0$), the bank charges the steady-state loan rate that compensates for its lending costs and expected default losses ($R = (1+c^B)R^{Def}$). In the steady state, The bank charges a positive credit spread reflecting expensive equity financing ($r^E > 0$) made necessary by the limit on leverage imposed by the binding VaR constraint.

PROPOSITION 8: *Assuming that the VaR constraint is binding near the steady state, then credit market clears at the level of bank leverage that is a unique solution to the equation below*

$$\frac{1}{\lambda_t^B} = 1 + c^B - g'(E_t^B\lambda_t^B)e^{-S_t^{VaR}}\{\mathbf{E}_t R_{t+1}^I/R_t^D\}. \tag{63}$$

Equation 63 implies that the following results hold near the steady state.

- *Given the level of bank equity* E_t^B *and the level of* S_t^{VaR}*, leverage increases with the level of demand for credit summarised by the variable* $\mathbf{E}_t R_{t+1}^I/R_t^D$*.*
- *Given the levels of* E_t^B *and* $\mathbf{E}_t R_{t+1}^I/R_t^D$*, leverage decreases with* S_t^{VaR}*.*
- *Given the levels of* S_t^{VaR} *and* $\mathbf{E}_t R_{t+1}^I/R_t^D$*, leverage decreases with the bank equity* E_t^B*.*

In addition, the equation below describes the dynamics of credit spreads

$$e^{S_t}\left\{g'\left(\frac{E_t^B}{1+c^B-e^{S_t-S_t^{VaR}}}\right)\right\}^{-1} = \frac{\mathbf{E}_t R_{t+1}^I}{R_t^D}. \tag{64}$$

Credit spreads increase with demand $\frac{\mathbf{E}_t R_{t+1}^I}{R_t^D}$ *and the VaR scenario credit spread* S_t^{VaR} *and decrease with* E_t^B*.*

PROOF:

See appendix C.C10.

The results of proposition 8 clarify the factors affecting the bank leverage dynamics in the model. These factors are the bank equity $E_t^B = (1 - d_{t-1})\pi_t^B$, which is determined by the performance of the bank's portfolio; the loan demand driven by the variable $\mathbf{E}_t R_{t+1}^I / R_t^D$; and the credit spread reflecting the riskiness of the VaR scenario S_t^{VaR}.[17] Ceteris paribus, leverage increases with loan demand and decreases with the level of bank equity and the VaR credit spread S_t^{VaR}. The overall impact of fluctuations on bank leverage depends on which effect prevails. After a shock that reduces bank equity more than future expectations of defaults and future returns on capital investment, the bank compensates for the loss of equity by increasing leverage, thus reducing the shock's impact on credit supply. On the other hand, a shock that increases future default expectations (thus increasing S^{VaR}) without a substantially increasing contemporaneous default that matters to the bank equity or a significantly increasing loan demand would tend to reduce leverage. Finally, shocks that only affect the expected return on capital investments would push the bank to accommodate the changes in loan demand using leverage as a control variable.

The credit spreads dynamics are intuitive. Credit spreads decrease when the bank balance sheet allows more lending (higher equity E_t^B) and when demand is lower (lower $\mathbf{E}_t R_{t+1}^I / R_t^D$) and increase when the bank adopts a more pessimistic VaR default scenario (higher S^{VaR}). This also means that credit spreads can be procyclical when shocks reduce demand for loans more than the bank equity and the bank's appetite for risk.

Note that the non-linearity of the capital production function is crucial for the bank's equity to have a direct effect on leverage. A linear capital production function ($g' = 1$) is a common assumption in economic literature.[18] Maintaining this assumption in the current set-up would mute the direct effects of the bank equity on leverage. Thus implying that leverage is always procyclical. Similarly, $g' = 1$ implies that credit spreads display the same cyclical behaviour as loan demand.

[17]See the first-order condition 12 for why demand for loans is driven by $\mathbf{E}_t R_{t+1}^I / R_t^D$.

[18]Firms or entrepreneurs linearly transforming capital with an a posteriori multiplicative idiosyncratic affecting capital production is a ubiquitous assumption in the macro-finance literature (e.g. Bernanke, Gertler and Gilchrist, 1998; Gourio, 2013; Nuño and Thomas, 2017, ...).

IV. Model simulations and findings

The model simulation considers the impact of potential future large shocks on risk premia through a VaR rule restricting the behaviour of financial intermediaries and ignores its effects on agents' expectations otherwise. The simulation is performed using second-order perturbation near the unique steady-state and estimated aims at matching several empirical first and second moments of real and financial variables at the annual frequency. The measure of the Value-at-Risk used in this section corresponds to the economic capital VaR introduced in definition 2 and studied analytically in subsection II.C, and exogenous shocks are limited to unexpected TFP fluctuations.

A. *Estimation and steady-state results*

The model is estimated to match several first and second moments of several RBC and financial sector aggregates. Following Christiano, Motto and Rostagno (2010), the preferences discounting parameter $\beta = 0.98$ is chosen so that the model's steady-state deposit rate matches the average historical deposit rates in the United States. The dislike for work parameter χ is chosen to match steady-state labour at $L = 0.33$.

The volatility parameter affecting the riskiness of capital production σ_K is important to the dynamics of default rates and the adjustment costs parameter κ_K is important to the persistence of default rates and thus to the variance of credit spreads. Through affecting credit markets, the parameters σ_K and κ_K affect the output's dynamics. The parameters $(\sigma_K, \kappa_K, \sigma^a, \rho^a)$ are then estimated for the model to match the historical standard deviations of the delinquency rate of commercial and industrial loans (C&I loans), BAA-AAA credit spreads, the combined assets of the aggregate financial intermediation sector in the U.S., the combined equity of the aggregate financial intermediation sector in the U.S. and output; the historical correlation of the aggregate financial sector's equity and with output; and the historical output's autocorrelation.[19] The steady-state distance to default given by $\xi = -\frac{\ln(\gamma_K)}{\sigma_K} - \frac{\sigma_K}{2}$ determines the steady-state default rate is $\Phi(-\xi)$. Given the value of σ_K, the parameter γ_K is calibrated to match the average delinquency rate of C&I loans at 2.6%. The loss in production following default θ_K is chosen to match the historical C&I loan charge-off average rates

[19] I use a simulated method of moments implemented in DYNARE.

at 0.75%. The likelihood of a large TFP shock $\mu_a = 2\%$ and their average size $J_a = 30\%$ are borrowed from Barro (2006), while the standard deviation of the shock size $\sigma_J = 30\%$ is borrowed from Gourio (2013). The steady-state credit spread matches the average historical net interest margin at $S = 4\%$. The VaR TFP shock scenario ϵ^{VaR} and the equity premium r^E are calibrated to match the steady-state credit spread at $S = 4\%$ and the steady-state leverage at $\lambda^B = 12.5\times$. The VaR probability corresponding to the distribution of TFP and the calibrated TFP shock ϵ^{VaR} is $p^{VaR} = 99\%$. The calibrated value of p^{VaR} is in line with values used in practice (Ball and Fang, 2006).

Table 2 summarizes the model parameters, while Table 3 lists key steady-state variables. Notably, the model's implied steady-state credit spread $S = 4\%$ significantly surpasses the credit spread from default risk alone, $S^{Def} = 0.81\%$. This excess premium encompasses both the bank's operational lending costs ($c^B = 2.5\%$) and the leverage limit due to the Value-at-Risk rule (the remaining 0.69%).

Additionally, Table 4 showcases the second moments of the primary variables. The model closely replicates targeted second moments except for the model's generated bank equity standard deviation, which is about half its empirical counterpart. The model-implied correlation between equity and output is close to zero with the corresponding empirical correlation being statistically negligible. The model-produced standard deviations of bank leverage and bank assets are close to the values observed in the data. In addition, the model reproduces the correct cyclical behaviour of bank assets, credit spreads, default rates and bank leverage, as shown by the signs of the correlations between these variables and output. However, the model dynamics imply higher correlations in absolute value. This is expected in a model where a single exogenous process (TFP) drives all other variables.

Increasing the VaR TFP shock scenario ϵ^{VaR} decreases the bank's insolvency probability (top-left panel of figure 3). However, a more resilient banking sector comes at the cost of higher financing costs for the productive sector (bottom panel of figure 3). Expensive financing of capital investments, in turn, decreases the steady-state output (top-right panel of figure 3). The bottom panel of figure 3 shows that the increase in financing costs caused by a more resilient banking sector is fully due to higher risk premia as the default risk among borrowing firms is unaffected by the bank's risk aversion. A more risk-averse banking sector decreases insolvency risk by adopting a more severe VaR scenario. This increases the VaR credit spread S^{VaR}, which, in turn, increases the risk premium S^{prem} and the charged credit spread S.

Figure 3 shows that reducing the bank's insolvency probability from 1.8% to 0.04% more than doubles the risk premium charged by banks (from 258bp to 597bp). The studied framework enables us to comment on the trade-offs between a more resilient banking sector and cheap funding for the productive sector. This framework, however, does not provide a way to quantify the welfare losses due to banks' insolvency. This means that we cannot make normative statements regarding the optimal level of insolvency risk from a macroprudential perspective.[20]

B. Dynamic effects

In this subsection, I show the dynamic effects of the model's main assumptions by studying the impulse response functions following negative shocks to the aggregate total factor productivity Z_t^a. The model simulations are realised at the annual frequency through perturbation techniques, using second-order approximations.

Figure 4 shows the impulse response function (IRF) of the model's main credit variables after one standard deviation unexpected negative shock to logarithmic TFP ($\ln Z^a$). The IRFs are shown for the main model as calibrated in section IV.A (red, continuous line) and a version of the model assuming constant leverage ($\lambda_t^B = \lambda^B$; blue, dashed lines). Except for the VaR-related parameters, both models are simulated using the same set of parameters in table 2. In addition, the constant leverage model assumes the same steady-state leverage as the main model. This guarantees that the steady-state risk premium is the same in the main model and the model with constant leverage. Clearly, the presence of the VaR constraint increases the reaction of credit spreads to real shocks. This, in turn, causes a larger drop in the loans dedicated to capital production in the presence of a Value-at-Risk constraint on bank lending. Fluctuations in credit spreads are larger under a VaR constraint because defaults are expected to be higher in the following period (top-right panel). This is consistent with expression 58 showing that the VaR default scenario fluctuates solely because of changes to default expectations.

Figure 4 also shows that immediately following the unexpected TFP shock, de-

[20]More steady-state results are presented in appendix D.D1.

fault rates are lower when banks are constrained by a VaR rule (top-left panel). The bottom panel of figure 4 helps explain this model feature. The figure shows that immediately following real shocks, the drop in the return on capital investment R^I is less pronounced when banks are constrained by a VaR rule. Proposition 1 shows that the immediate reaction of default rates to shocks is solely driven by changes in R^I. This links the moderate increase in defaults at the time of the shock under a VaR rule to the moderate drop in R^I. The difference in the behaviour of R^I results from capital being more scarce under a VaR rule where constrained lending reduces investment in new capital, thus increasing capital's marginal product.

The impulse responses of bank leverage, bank equity and bank lending are shown in the left panel of figure 5. The figure shows the impact of higher default rates on the bank's equity: as the default rate unexpectedly increases at the time of the shock, the bank's equity is lower. If leverage were fixed, the decrease of the bank equity would lead to the same relative drop in lending supply. Leverage, however, is procyclical. The procyclical behaviour of banks' leverage compounds the effects of lower bank equity leading to more constrained bank lending. In the period following the TFP shock, the bank can adjust its equity levels higher. This is made easier by higher risk premia that imply higher bank profits (right panel of figure 5). However, low bank leverage more than compensates for higher equity so bank lending remains below its steady-state level in the period following the shock. While fluctuations of expected defaults are the proximate reason for credit spread fluctuations, most of the variance of credit spread comes in the form of changes to credit premia. The simulation results are thus consistent with the empirical findings in Gilchrist and Zakrajšek (2012).

The procyclical behaviour of bank lending and bank leverage is in line with the stylised facts discussed in section I. The model also reproduces the empirically observed acyclical behaviour of bank equity. Equity decreases at the time of the shock before increasing as the staggered bank equity reaction takes place. This implies a low correlation between bank equity and output, as equity surpasses its steady-state value before output recovers. This explanation is consistent with the empirical behaviour of bank equity shown in figure 1, where the equity of the aggregate financial sector appears to decrease in the early stages of the Great Recession before recovering to higher levels in the later stages of the recession.

Figure 6 presents impulse responses following an adverse TFP shock when capital production is riskier ($\sigma_K = 0.1$) and not subject to adjustments costs ($\kappa_K = 0$). Higher σ_K implies higher default rates in the steady state, thus more fluctuation in the bank's equity due to higher than expected default rates. As discussed above, in the absence of adjustment costs, both the default risk credit spread and the VaR scenario credit spread remain stable (bottom panel of figure 6). In line with the results of proposition 8, the larger impact of the TFP shock on the bank's equity through losses in the bank's lending portfolio and the absence of fluctuations in future default expectations imply countercyclical leverage (top panel of figure 6). The results can also be read in light of the leverage constraint 40, as higher credit spreads S and stable VaR credit spreads S^{VaR} also mean higher and thus countercyclical leverage.

V. Concluding remarks

I present a simple general equilibrium model with endogenous defaults and a financial intermediation sector subject to a Value-at-Risk rule. While default rates in the model are too low to generate quantitatively realistic credit spreads, the expectation of future default losses interacts with the financial intermediary's Value-at-Risk rule, generating an empirically plausible risk premium in the steady state. In addition, the model generates quantitatively realistic credit spread dynamics, mainly driven by the fluctuations of risk premia. The fluctuations in credit spreads are generated by changes in credit supply as bank leverage is constrained by a Value-at-Risk rule and frictions in equity financing prevent banks from immediately replenishing their equity capital. Moreover, the model generates fluctuations in the financial sector's balance sheet that are consistent with empirical evidence. Namely, a procyclical supply of credit that is mainly driven by procyclical fluctuations of leverage in the financial sector. The impact of the model's Value-at-Risk mechanisms on credit supply amplifies business cycle fluctuations. In the presence of this mechanism, bank lending is more volatile, leading to more volatile aggregate investments. The model's relative simplicity and analytical tractability make it easily extendable in ways that can shed light on monetary policy, macroprudential policy and their areas of interaction.

REFERENCES

Adrian, Tobias, and Fernando Duarte. 2018. "Financial vulnerability and monetary policy."

Adrian, Tobias, and Hyun Song Shin. 2010. "Liquidity and leverage." *Journal of Financial Intermediation*, 19(3): 418–437.

Adrian, Tobias, and Hyun Song Shin. 2014. "Procyclical Leverage and Value-at-Risk." *Review of Financial Studies*, 27(2): 373–403.

Arellano, Cristina, Yan Bai, and Patrick J Kehoe. 2019. "Financial frictions and fluctuations in volatility." *Journal of Political Economy*, 127(5): 2049–2103.

Ball, Jason, and Victor Fang. 2006. "A Survey of Value-at-Risk and its Role in the Banking Industry." *Journal of Financial Education*, 32: 1–31.

Barro, Robert J. 2006. "Rare Disasters and Asset Markets in the Twentieth Century." *The Quarterly Journal of Economics*, 121(3): 823–866.

Bernanke, Ben, Mark Gertler, and Simon Gilchrist. 1998. "The Financial Accelerator in a Quantitative Business Cycle Framework." National Bureau of Economic Research Working Paper 6455.

Brei, Michael, and Leonardo Gambacorta. 2016. "Are bank capital ratios pro-cyclical? New evidence and perspectives." *Economic Policy*, 31(86): 357–403.

Carlstrom, Charles T., and Timothy S. Fuerst. 1997. "Agency Costs, Net Worth, and Business Fluctuations: A Computable General Equilibrium Analysis." *The American Economic Review*, 87(5): 893–910.

Christiano, Lawrence, Roberto Motto, and Massimo Rostagno. 2010. "Financial factors in economic fluctuations." *Working Papers Series, European Central Bank*, 47(1).

Cohen-Cole, Ethan, Burcu Duygan-Bump, Jose L Fillat, and Judit Montoriol-Garriga. 2008. "Looking Behind the Aggregates: A Reply to 'Facts and Myths about the Financial Crisis of 2008'." *FRB of Boston Quantitative Analysis Unit Working Paper*, 8(5).

Geanakoplos, John. 2010. "The leverage cycle." *NBER macroeconomics annual*, 24(1): 1–66.

Gertler, Mark, and Nobuhiro Kiyotaki. 2010. "Chapter 11 - Financial Intermediation and Credit Policy in Business Cycle Analysis." In . Vol. 3 of *Handbook of Monetary Economics*, , ed. Benjamin M. Friedman and Michael Woodford, 547–599. Elsevier.

Gertler, Mark, and Peter Karadi. 2011. "A model of unconventional monetary policy." *Journal of Monetary Economics*, 58(1): 17–34. Carnegie-Rochester Conference Series on Public Policy: The Future of Central Banking April 16-17, 2010.

Gilchrist, Simon, and Egon Zakrajšek. 2012. "Credit Spreads and Business Cycle Fluctuations." *American Economic Review*, 102(4): 1692–1720.

Gomes, Joao F., Amir Yaron, and Lu Zhang. 2003. "Asset prices and business cycles with costly external finance." *Review of Economic Dynamics*, 6(4): 767 – 788. Finance and the Macroeconomy.

Gourio, François. 2012. "Disaster Risk and Business Cycles." *American Economic Review*, 102(6): 2734–66.

Gourio, François. 2013. "Credit Risk and Disaster Risk." *American Economic Journal: Macroeconomics*, 5(3): 1–34.

Holmström, Bengt, and Jean Tirole. 1998. "Private and Public Supply of Liquidity." *Journal of Political Economy*, 106(1): 1–40.

Kiyotaki, Nobuhiro, and John Moore. 1997. "Credit Cycles." *Journal of Political Economy*, 105(2): 211–248.

Merton, Robert C. 1974. "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates." *The Journal of Finance*, 29(2): 449–470.

Nuño, Galo, and Carlos Thomas. 2017. "Bank Leverage Cycles." *American Economic Journal: Macroeconomics*, 9(2): 32–72.

Rietz, Thomas A. 1988. "The equity risk premium a solution." *Journal of Monetary Economics*, 22(1): 117–131.

Weil, Philippe. 1989. "The equity premium puzzle and the risk-free rate puzzle." *Journal of Monetary Economics*, 24(3): 401–421.

Appendix For Online Publication

Data Appendix

B1. Data sources and treatment

Balance sheet data of the aggregate financial sector is obtained from the equivalent data for the four subsectors (US-chartered commercial banks, savings institutions, security brokers and dealers, and finance companies) for which data is available in the Z.1 files of the U.S. Flow of Funds. Correction for discontinuity in the balance sheet time series is applied following the methodology outlined in the online appendix of Nuño and Thomas (2017).

The net interest rate margin time series is from the Federal Financial Institutions Examination Council (USNIM on FRED). The constant lending cost is deduced from the average Bank's Non-Interest Income to Total Income for the United States for which the World Bank provides annual data between 2000 and 2020 (DDEI03USA156NWDB on FRED).

GDP and investment data are from the Bureau of Economic Analysis. All cyclical time series have been detrended using a Hodrick-Prescott filter, with a smoothing parameter of 1600 for quarterly data and 100 for annual data.

Following Gourio (2013), I show in figure B1 the time series of the difference between the yields of BAA-rated and AAA-rated bonds by Moody's plotted against aggregate investment in the United States. The BAA-AAA spread can be interpreted as a credit spread attached to BAA-rated bonds, as the AAA-rated securities are virtually risk-free. Credit spreads appear to be countercyclical, and as expected by economic theory, are negatively correlated with aggregate investments.

Mathematical Appendix

C1. A useful lemma

LEMMA 1: *If ϵ is a normal variable and A and $S > 0$ are two real numbers, then*

$$\int_{\epsilon=-\infty}^{A} e^{S\epsilon - S^2/2} d\Phi(\epsilon) = \Phi(A - S). \tag{C1}$$

PROOF:

I write the integral in the lemma by replacing with the density of the normal variable

$$\int_{\epsilon=-\infty}^{A} e^{S\epsilon - S^2/2} d\Phi(S) = \frac{1}{\sqrt{2\pi}} \int_{\epsilon=-\infty}^{A} e^{S\epsilon - S^2/2} e^{-\epsilon^2/2} d\epsilon$$
$$= \frac{1}{\sqrt{2\pi}} \int_{\epsilon=-\infty}^{A} e^{-(\epsilon - S)^2/2} d\epsilon$$

Through the change of variable $\epsilon' = \epsilon - A$

$$\int_{\epsilon=-\infty}^{A} e^{S\epsilon - S^2/2} d\Phi(S) = \frac{1}{\sqrt{2\pi}} \int_{\epsilon=-\infty}^{A-S} e^{-(\epsilon')^2/2} d\epsilon$$
$$= \Phi(A - S).$$

C2. Aggregate recovery expression

From 11 the firm i revenues from selling the produced final goods and produced at time t capital are

$$R_t^I e^{\sigma_K \epsilon_i - \sigma_K^2/2} g(X_{t-1}). \tag{C2}$$

The defaulting firms, i.e. those with an idiosyncratic shock lower than $-\xi_t$ have the aggregate revenues

$$(1 - \theta_K) R_t^I g(X_{t-1}) \int_{\epsilon_i < -\xi_t} e^{\sigma_K \epsilon_i - \sigma_K^2/2}. \tag{C3}$$

Using the result in lemma 1 in the appendix, the amount recovered by banks is

$$Rec_t = (1 - \theta_K) R_t^I g(X_{t-1}) \Phi(-\xi_t - \sigma_K). \tag{C4}$$

The aggregate produced capital at each period is reduced by the amount lost to bankruptcy workouts

$$\text{(C5)}\quad I_t = g(X_{t-1}) \underbrace{\int_{\epsilon=-\infty}^{+\infty} e^{\sigma_K \epsilon - \sigma_K^2/2} d\Phi(\epsilon)}_{=1} - \theta_K g(X_{t-1}) \underbrace{\int_{\epsilon=-\infty}^{-\xi_t} e^{\sigma_K \epsilon - \sigma_K^2/2} d\Phi(\epsilon)}_{=\Phi(-\xi_t - \sigma_K)},$$

where the second term in the rhs represents the loss of output due to defaults. Rewrite using the lemma

$$\text{(C6)}\quad I_t = g(X_{t-1}) \left[1 - \theta_K \Phi(-\xi_t - \sigma_K)\right].$$

Replace for $R_t^I g(X_{t-1})$ using the definition of distance to default 13 to get an alternative recovery expression

$$\text{(C7)}\quad Rec_t = (1-\theta_K) R_{t-1} X_{t-1} e^{\sigma_K \xi_t + \sigma_K^2/2} \Phi(-\xi_t - \sigma_K).$$

C3. VaR expression

Write the bank's Value-at-Risk, using the aggregate recovery expression 20 and the borrowing firm first-order condition

$$\begin{aligned} VaR_t &= (1+c^B)X_t - \frac{R_t}{R_t^D} X_t \Phi(\xi_t^{VaR}) - \frac{1}{R_t^D} Rec_t^{VaR} \\ &= X_t \left\{ 1 + c^B - \frac{R_t}{R_t^D} \left[\Phi(\xi_t^{VaR}) + (1-\theta_K) e^{\sigma_K \xi_t^{VaR} + \sigma_K^2/2} \Phi(-\xi_t^{VaR} - \sigma_K) \right] \right\} \\ &= X_t \left\{ 1 + c^B - e^{S_t - S_t^{VaR}} \right\}, \end{aligned}$$

where by definition

$$\text{(C8)}\quad S_t^{VaR} := -\ln\left(\Phi(\xi_t^{VaR}) + (1-\theta_K) \mathbf{e}^{\sigma_K \xi_t^{VaR} + \sigma_K^2/2} \Phi(-\xi_t^{VaR} - \sigma_K) \right).$$

C4. Proof of proposition 1

The loan demand first-order condition writes

(C9)

$$\frac{R_t X_t}{\mathbf{E}_t[R_{t+1}^I]} = \gamma_K X_t^{\gamma_K} \left[1 - \kappa_K \left\{ (X_t/X) - 1 \right\}^2 \right] - 2\kappa_K X_t^{\gamma_K} (X_t/X) \{X_t/X - 1\}.$$

Substitute into the distance-to-default definition 13 to obtain

(C10)

$$\sigma_K \xi_{t+1} = -\ln(\gamma_K) - \frac{\sigma_K^2}{2} + \ln\left(\frac{R^I_{t+1}}{\mathbf{E}_t R^I_{t+1}}\right) - \ln\left(1 - \frac{2\kappa_K}{\gamma_K} \frac{(X_t/X)\{X_t/X - 1\}}{1 + \kappa_K\{X_t/X - 1\}^2}\right).$$

The proposition's steady-state results obtains for $X_t = X$ and the dynamics' result from deducting the steady-state distance-to-default from C10.

C5. Proof of proposition 2

When both the leverage and the budget constraints are binding, the deposits D_t are given by

$$D_t = \left\{1 + c^B - VaR_t^U\right\} X_t. \tag{C11}$$

From expression 20, the recovery Rec_t is given by

$$\frac{Rec_t}{R_{t-1}X_{t-1}} = (1 - \theta_K)e^{\sigma_K \xi_t + \sigma_K^2/2}\Phi(-\xi_t - \sigma_K). \tag{C12}$$

Replace for the recovery using C12 and for deposits D_t using $D_t = (1+c^B)X_t - E_t^B$

(C13)

$$\pi^B_{t+1} = R_t X_t \left\{\Phi(\xi_{t+1}) + (1 - \theta_K)e^{\sigma_K \xi_{t+1} + \sigma_K^2/2}\Phi(-\xi_{t+1} - \sigma_K) - (1 + c^B)e^{-S_t}\right\}$$
$$+ R_t^D (1 - d_{t-1})\pi_t^B.$$

Replace using $X_t = \frac{E_t^B}{VaR_t^U} = \frac{(1-d_{t-1})\pi_t^B}{VaR_t^U}$ in C13 to get the bank profit function at the Value-at-Risk constraint

(C14)

$$\pi^B_{t+1} = R_t(1 - d_{t-1})\frac{\pi_t^B}{VaR_t^U}$$
$$\left\{\Phi(\xi_{t+1}) + (1 - \theta_K)e^{\sigma_K \xi_{t+1} + \sigma_K^2/2}\Phi(-\xi_{t+1} - \sigma_K) - (1 + c^B - VaR_t^U)e^{-S_t}\right\}.$$

One can rewrite the bank's problem using the profit expression C14 that uses the bank's budget contract to replace for deposits and the binding VaR constraint to replace for the loan size. The bank's problem thus becomes

$$\max_{d_s} \mathbf{E}_t \sum_{s \geq t+1} \frac{m_{t,s}}{(1+r_t^E)^{s-t}} d_{s-1} \pi_s^B \tag{C15}$$

subject to the repeated constraint on d_s

$$\text{s.t. } d_s \leq 1, \tag{C16}$$

and where the bank profit π_s^B is given by expression C14. From expression C14 and given the assumption that the bank neglects the effect of its lending on default probabilities and the VaR linearity assumption

$$\frac{\partial \pi_{t+2}^B}{\partial d_t} = -\frac{\pi_{t+2}^B}{1-d_t} \tag{C17}$$

and

$$\frac{\partial \pi_{t+s}^B}{\partial d_t} = \frac{\pi_{t+s}^B}{\pi_{t+s-1}^B} \frac{\partial \pi_{t+s-1}^B}{\partial d_t} \text{ for } s \geq 3. \tag{C18}$$

By iteration

$$\frac{\partial \pi_{t+s}^B}{\partial d_t} = -\frac{\pi_{t+s}^B}{1-d_t} \text{ for } s \geq 2. \tag{C19}$$

Using the last expression and $\frac{\partial d_t \pi_{t+1}^B}{\partial d_t} = \pi_{t+1}^B$, the bank's problem first-order condition thus yields

$$\mathbf{E}_t \frac{m_{t,t+1}}{1+r^E} E_{t+1}^B = \mathbf{E}_t \frac{m_{t,t+1}}{1+r^E} V_{t+1}^B. \tag{C20}$$

where V_t^B designates the bank's value $V_t^B = \mathbf{E}_t \sum_{s \geq t+1} \frac{m_{t,s}}{1+r^E} d_{s-1} \pi_s^B$.

C6. Lending when the VaR rule is not binding

The budget constraint yields the deposit expression below

$$D_t = (1+c^B) X_t - (1-d_{t-1}) \pi_t^B. \tag{C21}$$

Plug into the definition of the bank profit along with the expression of Rec_t in 20

(C22)
$$\pi_{t+1}^B = R_t X_t \left\{ \Phi(\xi_{t+1}) + (1-\theta_K) e^{\sigma_K \xi_{t+1} + \sigma_K^2/2} \Phi(-\xi_{t+1} - \sigma_K) - \frac{R_t^D}{R_t}(1+c^B) \right\} + R_t^D (1 - d_{t-1}) \pi_t^B.$$

In the absence of the Value-at-Risk constraint, rewrite the bank's problem

$$\max_{X_s, d_s} \mathbf{E}_t \sum_{s \geq t} \frac{m_{t,s}}{(1+r_t^E)^{s-t}} d_{s-1} \pi_s^B. \tag{C23}$$

subject to the repeated constraint $d_s \leq 1$, and where the bank profit π_s^B is given by expression C22. Deriving with regard to d_t

$$\mathbf{E}_t m_{t,t+1} \pi_{t+1}^B - \mathbf{E}_t \frac{m_{t,t+2}}{1+r_t^E} R_{t+1}^D d_t \pi_{t+2}^B \tag{C24}$$

The derivative above is positive near the steady state, implying a full distribution of profits when leverage is unconstrained ($d_t = 1$). The first-order condition with regard to X_t writes

$$R_t \mathbf{E}_t \left\{ \Phi(\xi_{t+1}) + (1-\theta_K) e^{\sigma_K \xi_{t+1} + \sigma_K^2/2} \Phi(-\xi_{t+1} - \sigma_K) \right\} = R_t^D (1+c^B), \tag{C25}$$

or $R_t = (1+c^B) R_t^{Def}$.

C7. Proof of proposition 3

First, note the steady-state deposit rate follows immediately from the households' Euler saving equation $R^D = 1/\beta$. From the expression of the bank's value

$$V^B = \frac{\beta}{1 + r^E - \beta} d\pi^B. \tag{C26}$$

The steady-state equity is given by

$$E^B = (1-d)\pi^B. \tag{C27}$$

Plug into the steady-state version of the first-order condition 47 to get

$$(1-d)\pi^B = \frac{\beta}{1+r^E-\beta} d\pi^B. \tag{C28}$$

Rearrange

$$d = 1 - \frac{\beta}{1+r^E}. \tag{C29}$$

Write the Value-at-Risk constraint again, replacing for π^B and for D using C14

$$\begin{aligned} &VaR^U X = (1-d)X \\ &\left[R\left\{\Phi(\xi) + (1-\theta_K)e^{\sigma_K\xi+\sigma_K^2/2}\Phi(-\xi-\sigma_K)\right\} - R^D(1+c^B-VaR^U)\right] \end{aligned} \tag{C30}$$

Replace for d using C29 and rearrange using the definition of the credit spread $R = e^S/\beta$

$$r^E VaR^U = e^S\left\{\Phi(\xi) + (1-\theta_K)e^{\sigma_K\xi+\sigma_K^2/2}\Phi(-\xi-\sigma_K)\right\} - (1+c^B) \tag{C31}$$

Using the definition of S^{Def}

$$r^E VaR^U = e^{S-S^{Def}} - (1+c^B). \tag{C32}$$

Rearrange

$$e^S = e^{S^{Def}}\left(1 + c_B + r^E VaR^U\right). \tag{C33}$$

C8. Proof of proposition 4

Replacing for for Rec_{t+1} using expression 20, he bank's profit is

$$\begin{aligned} &\pi^B_{t+1} = R_t X_t\left[\Phi(\xi_{t+1}) + (1-\theta_K)e^{\sigma_K\xi_{t+1}+\sigma_K^2/2}\Phi(-\xi_{t+1}-\sigma_K)\right] \\ &- R^D_t(1+c^B)X_t + R^D_t E^B_t. \end{aligned} \tag{C34}$$

Noting that the bank neglects the impact of its marginal new lending on default probabilities, one can write the first derivative of the bank profits with regard to the loan size X_t as follows

(C35)
$$\frac{\partial \pi^B_{t+1}}{\partial X_t} = R_t \left\{ \Phi(\xi_{t+1}) + (1-\theta_K) e^{\sigma_K \xi_{t+1} + \sigma_K^2/2} \Phi(-\xi_{t+1} - \sigma_K) \right\} - R^D_t (1 + c^B).$$

Clearly, $\frac{\partial \pi^B_{t+1}}{\partial X_t} \geq 0$ if and only if $R_t \geq (1+c^B) R^{Def}_t$. Moreover, the case $R_t < (1+c^B) R^{Def}_t$ can be excluded as the bank would not find it optimal to lend in this case. One can, therefore, conclude that $R_t \geq (1+c^B) R^{Def}_t$.

Assuming $R_t = (1+c^B) R^{Def}_t$, the VaR constraint binds when credit supply reaches the limit

(C36)
$$X_t = \frac{E^B_t}{VaR^U((1+c^B)R^{Def}_t, \mathcal{D}(\xi_{t+1}))}.$$

When $R_t > (1+c^B) R^{Def}_t$ The VaR constraint writes

(C37)
$$X_t = \frac{E^B_t}{VaR^U(R_t, \mathcal{D}(\xi_{t+1}))},$$

thus determining the level of lending supply. This finishes the proposition's proof.

C9. Proof of proposition 5

Write the definition of ξ^{VaR}_t and replace for Q^{VaR}_t using assumption 3.

$$\begin{aligned} \xi^{VaR}_t &= \frac{1}{\sigma_K} \ln \left(\frac{Q^{VaR}_t g(X_t)}{R_t X_t} \right) - \frac{\sigma_K}{2} \\ &= \frac{1}{\sigma_K} \ln(\mathbf{E}_t Q_{t+1}) + \frac{1}{\sigma_K} \ln \left(\frac{g(X_t)}{R_t X_t} \right) - \frac{\sigma_K}{2} - \frac{\alpha_{R^I}}{\sigma_K} \epsilon^{VaR}. \end{aligned}$$

From the definition of ξ_{t+1}

(C38)
$$\mathbf{E}_t Q_{t+1} = \frac{R_t X_t}{g(X_t)} \mathbf{E}_t e^{\sigma_K \xi_{t+1} + \sigma_K^2/2}.$$

Plug in the expression of ξ^{VaR}_t above

(C39)
$$\xi^{VaR}_t = \frac{1}{\sigma_K} \ln \left(\mathbf{E}_t e^{\sigma_K \xi_{t+1}} \right) - \frac{\alpha_{R^I}}{\sigma_K} \epsilon^{VaR}.$$

C10. Proof of proposition 8

Combine the VaR constraint with the loan demand condition

$$\frac{1}{\lambda_t^B} = 1 + c^B - g'(X_t)\frac{\mathbf{E}_t R_{t+1}^I}{R_t^{VaR}}. \tag{C40}$$

Replace for X_t using $X_t = \lambda_t^B E_t^B$ and for R_t^{VaR} using $R_t^{VaR} = R_t^D e^{S_t^{VaR}}$

$$\frac{1}{\lambda_t^B} = \underbrace{1 + c^B - g'(E_t^B \lambda_t^B) e^{-S_t^{VaR}} \frac{\mathbf{E}_t R_{t+1}^I}{R_t^D}}_{:=\mathcal{H}_t(\lambda_t^B)}. \tag{C41}$$

Now, note that $g'(X)$ is decreasing in X near the steady state.

(C42)

$$g'(X_t) = \gamma_K X_t^{\gamma_K - 1}\left[1 - \kappa_K\left\{(X_t/X) - 1\right\}^2\right] - 2\kappa_K X_t^{\gamma_K}(1/X)\{X_t/X - 1\}$$
$$\approx \gamma_K X_t^{\gamma_K - 1}.$$

Hence

$$\begin{aligned} g''(X) &= -\gamma_K(1-\gamma_K)X_t^{\gamma_K-2}\left[1 - \kappa_K\left\{(X_t/X) - 1\right\}^2\right] \\ &- 4\gamma_K\kappa_K X_t^{\gamma_K-1}(1/X)\{X_t/X - 1\} - 2\kappa_K X_t^{\gamma_K}(1/X^2) \\ &\approx -\{\gamma_K(1-\gamma_K) + 2\kappa_K(X_t^2/X^2)\}X_t^{\gamma_K-2}. \end{aligned} \tag{C43}$$

As long as the bank's equity remains positive, the left hand side of C41 is decreasing in λ_t^B and the right hand side of the same equation, noted $\mathcal{H}_t(\lambda_t^B)$, is increasing in λ_t^B. Given the limits of the two sides of equation C41 for λ_t^B near zero and for very large value of λ_t^B, credit supply meets demand at a unique equilibrium leverage λ_t^B. Given the values of E_t^B and S_t^{VaR}, $\mathcal{H}_t$ becomes smaller for all values of λ_t^B when $\mathbf{E}_t R_{t+1}^I / R_t^D$ is higher. This means that for given values of E_t^B and S_t^{VaR}, the equilibrium bank leverage increases with $\mathbf{E}_t R_{t+1}^I / R_t^D$. Similarly, for given values of E_t^B and $\mathbf{E}_t R_{t+1}^I$, the equilibrium bank leverage decreases with R_t^{VaR}. Finally, for higher values of the bank equity, the application $\lambda_t^B \to g'(E_t^B \lambda_t^B)$ is lower all values of λ_t^B, meaning that $\mathcal{H}_t$ becomes larger for all values of λ_t^B. We can therefore conclude that for given values of S_t^{VaR} and $\mathbf{E}_t R_{t+1}^I / R_t^D$, the equilibrium bank leverage decreases with the bank equity E_t^B.

To get the dynamics of credit spreads replace using the binding VaR constraint in the loan demand condition and rearrange

$$e^{S_t}\left\{g'\left(\frac{E_t^B}{1+c^B-e^{S^t-S_t^{VaR}}}\right)\right\}^{-1}=\frac{\mathbf{E}_tR_{t+1}^I}{R_t^D}. \quad \text{(C44)}$$

As argued above, g' is decreasing near the steady state. This means that the RHS of C44 is increasing S_t and E_t^B and decreasing in S_t^{VaR}. This means that the credit spread increases with demand $\frac{\mathbf{E}_tR_{t+1}^I}{R_t^D}$ and the VaR scenario credit spread S_t^{VaR} and decreases with E_t^B.

Additional simulation results

D1. *Additional steady-state results*

Figure D1 shows the impact of the riskiness of capital production σ_K on the steady-state model's variables. The bottom panel shows that higher σ_K increases the steady-state credit spread S, which in turn decreases the steady-state demand for loans. Lower financing of capital production in the steady state leads to lower steady-state output (top-left panel). The increase in steady-state credit spreads is driven by higher steady-state default rates implying higher S^{Def}, as the steady-state credit spread premium S^{Prem} remains stable. The credit spread reflecting the riskiness of the VaR scenario S^{VaR} also increases with σ_K in the steady state. The increase in the steady-state credit spread S is outpaced by the increase in the VaR credit spread S^{VaR}. As per expression 40, this means that leverage decreases with σ_K. Finally, the steady-state bank equity E^B increases with σ_K. This is a result of the behaviour of the steady-state leverage and lending ($E^B = X/\lambda^B$).

It is worth noting that while the changes to σ_K have substantial effects on the steady-state credit spreads, the impact on loan financing and output is less significant. The steady-state credit spread increases from $S = 256bp$ when $\sigma_K = 1.4\%$ to $S = 725bp$ when the riskiness of capital production increases $\sigma_K = 9\%$. On the other hand, loan financing and output only decrease by 3.9% and 2.3% respectively when σ_K increases from 1.4% to 9%. This can be explained by the increase of the steady-state capital price Q that accompanies the increase in the cost of financing. Q increases by 4.4% and this increase helps compensate for the increase in the cost of lending from the perspective of the firm.[21] The changes in Q reflect changes in the rental of capital over its lifetime. Any change in the rental cost of capital $R^K = \alpha Y/K$ causes a larger change in Q.[22] This moderating effect of the price of capital also operates in the comparative statics of the equity risk premium r^E and the VaR TFP shock scenario ϵ^{VaR} presented below.

Figure D2 shows that the bank equity risk premium r^E does not impact steady-state default risk (S^{Def}) nor does it impact the riskiness of the VaR scenario (S^{VaR}). However, higher values of r^E increase the impact of the VaR scenario on credit supply thus increasing the steady-state credit spread S (see expression 62). Higher steady-state credit spreads due to higher equity premia r^E lower demand for loans in the steady state. This means lower steady-state capital production and output. The riskiness of the VaR scenario (S^{VaR}) is unaffected by equity risk premia, while the steady-state credit spread S increases with r^E. This implies that leverage increases with r^E (equation 40).

The steady-state effects of the assumed VaR TFP shock scenario ϵ^{VaR} are shown in figure D3. Higher values of ϵ^{VaR} increase the riskiness of the VaR scenario, increasing the steady-state values of S^{VaR}. This, in turn, increases the steady-state credit spread S, through higher risk premia S^{Prem}, while the credit spreads reflecting default risk S^{Def} do not change much with ϵ^{VaR}. The increase in S^{VaR} with ϵ^{VaR} is much larger than the increase in S. As a result, leverage decreases with ϵ^{VaR}. The bank reacts to more stringent leverage limits by increasing equity. Nonetheless, the increase in credit spreads S with ϵ^{VaR} lowers the demand for loans, thus decreasing capital production and output.

D2. Additional IRF simulations

Costly adjustments to capital production, captured by the parameter κ_K, play an important role in generating the dynamics of credit spreads. Figure D4 shows the impulse responses of leverage, bank equity, loans and credit spreads following an adverse TFP shock without adjustment costs ($\kappa_K = 0$). The figure shows that in the absence of adjustment costs, default expectations do not fluctuate. This is reflected by the muted response of the fundamental credit spreads S^{Def} and

[21]See the demand for financing equation 12.

[22]The steady-state price of capital Q moves multiple folds the movement in the rental price of capital $R^K = \alpha Y/K$. From the steady-state version of the household's capital first-order equation, and replacing for the calibrated model parameters: $Q = \frac{\beta}{1-\beta(1-\delta)} R^K \approx 8.3 \times R^K$.

VaR Credit spreads S^{VaR} (right panel). However, both credit spreads and credit premia drop, indicating that the shock impacts credit supply less than demand. In the absence of adjustment costs default expectations do not deteriorate and credit supply remains high relative to the main calibration case. On the other hand, leverage, bank equity and lending display procyclical behaviour at the time of the unexpected shock (left panel). The immediate reaction of bank equity is a consequence of the unexpected shock decreasing the return on investment R^I and thus increasing defaults (see equation 43). Leverage is lower because the credit spread S is lower while the VaR credit spread S^{VaR} remains stable (see the leverage constraint 40). The absence of adjustment costs also implies that the reaction of the credit spread S displays no persistence. This and the lack of fluctuation of the VaR credit spread S^{VaR} imply that leverage goes back to its steady-state value in the period following the shock and the fluctuations of credit supply are fully driven by the fluctuations of bank equity.

TABLE 2—ASSUMED AND CALIBRATED MODEL PARAMETERS.

Model parameter	**Value**	**Source/target**
Households preferences		
σ_H risk aversion	1	macro. literature
η_H labour disutility curvature	1	macro. literature
β discount factor	0.98	macro. literature
χ_H disutility of labour	8.0	$L = 0.3$
Technology		
α capital share	0.33	macro. literature
δ depreciation rate	10%	macro. literature
TFP shocks		
ρ_a shocks' persistence	0.48	SMM
σ_a log TFP volatility	1.15%	SMM
μ_a large shock probability	2%	Barro (2006)
$\bar{J}$ large shock size	30%	Barro (2006)
σ_J large shock std dev.	30%	Gourio (2013)
Capital production		
θ_K loss upon default	29%	steady-state credit losses 0.7%
γ_K curvature parameter	0.85	$\Phi(-\xi) = 2.6\%$
σ_K idiosyncratic risk	5.77%	SMM
κ_K adjustment cost	0.135	SMM
Banks		
c^B lending cost	2.5%	U.S. banks data
r^E equity premium	8.4%	{ r^E and ϵ^{VaR} target
ϵ^{VaR} VaR TFP shock	32%	the steady-state credit spread at 4% and leverage at 8×}
p^{VaR} VaR quantile	99.04%	{ Matches the value of ϵ^{VaR} given J_a and μ_a}
α_{R^I} $\ln(Z^a)$ to $\ln(R^I)$ coefficient	0.35	matches model dynamics

Note: SMM refers to the simulated method of moments used to determine the set of parameters $(\sigma_K, \kappa_K, \sigma^a, \rho^a)$.

Table 3—Steady-state variables.

Variable	**Steady state**
Debt level X	0.129
Output Y	0.571
Consumption C	0.439
Capital K	1.74
Investment I	0.174
Labour L	0.33
Default Rate $\Phi(-\xi)$	2.6%
Distance to default ξ	1.94
Credit spread S	4%
Defaults-implied credit cpread S^{Def}	0.81%
Credit Spread Premium S^{Prem}	3.19%
VaR credit spread S^{VaR}	9.6%
Leverage λ^B	12.5×
Unit VaR	0.08

Table 4—Second moments of log variables.

Variable	**Std. dev.**		**Y correlation**	
	Data	**model**	**Data**	**Model**
Loans	5.5%	**5.2%**	49%	93%
Default rates	32%	**32%**	-41%	-61%
Credit spreads	33%	**31%**	-19%	-89%
Bank equity	4.0%	**2.1%**	7.7%	**-1.4%**
Bank leverage	6.4%	5.1%	49%	94%
	Std. dev.		**Autocorrelation**	
	Data	model	Data	Model
Output	1.7%	**1.8%**	47%	**47%**

Note: standard deviations of the main model variables and their correlations with output. The autocorrelation of output is shown as it is a calibration target. All variables are in log form. All the model's moments targeted by the SMM routine are presented in bold in the table.

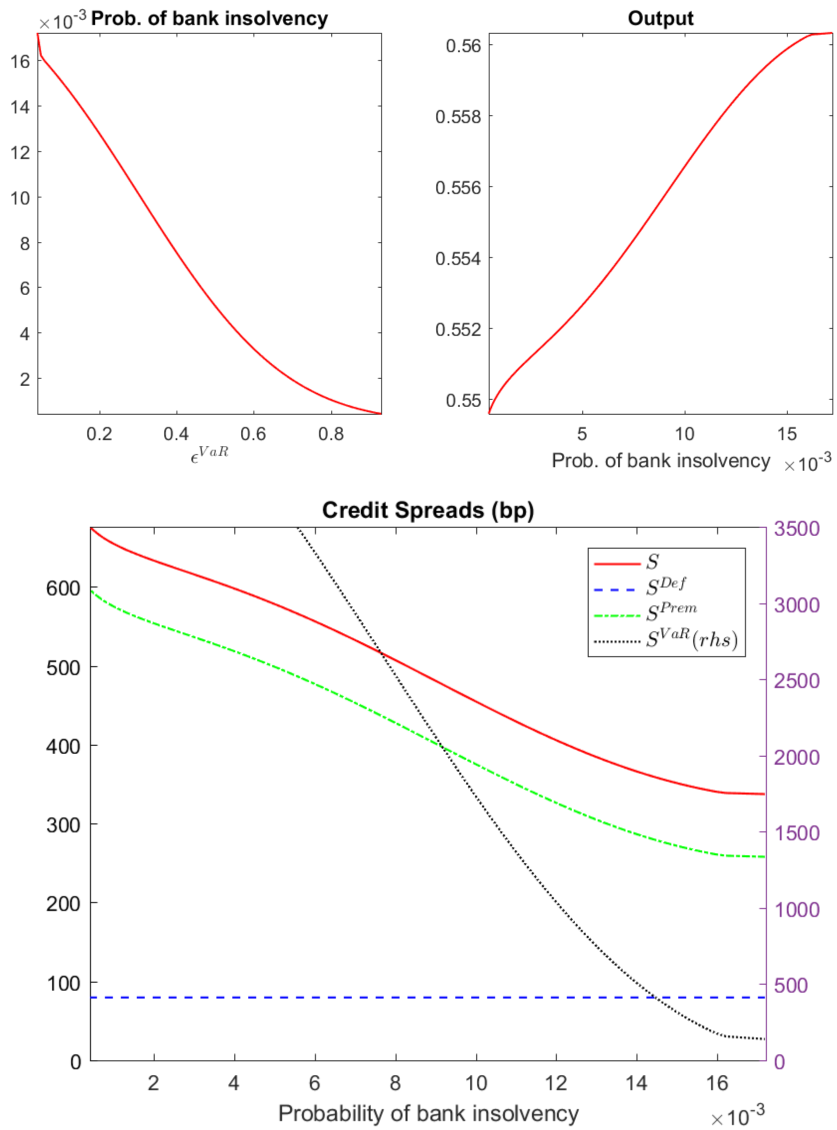

FIGURE 3. MACROPRUDENTIAL TRADEOFF: BANKS' SOLVENCY, FINANCING COSTS AND OUTPUT.

Note: The effect of increasing the bank's targeted insolvency probability on the steady-state values of output Y, the credit spread S, the default risk credit spread S^{Def}, the credit risk premium S^{Prem} and the VaR scenario credit spread S^{VaR} (rhs axis). The remaining model variables are the same as in table 2.

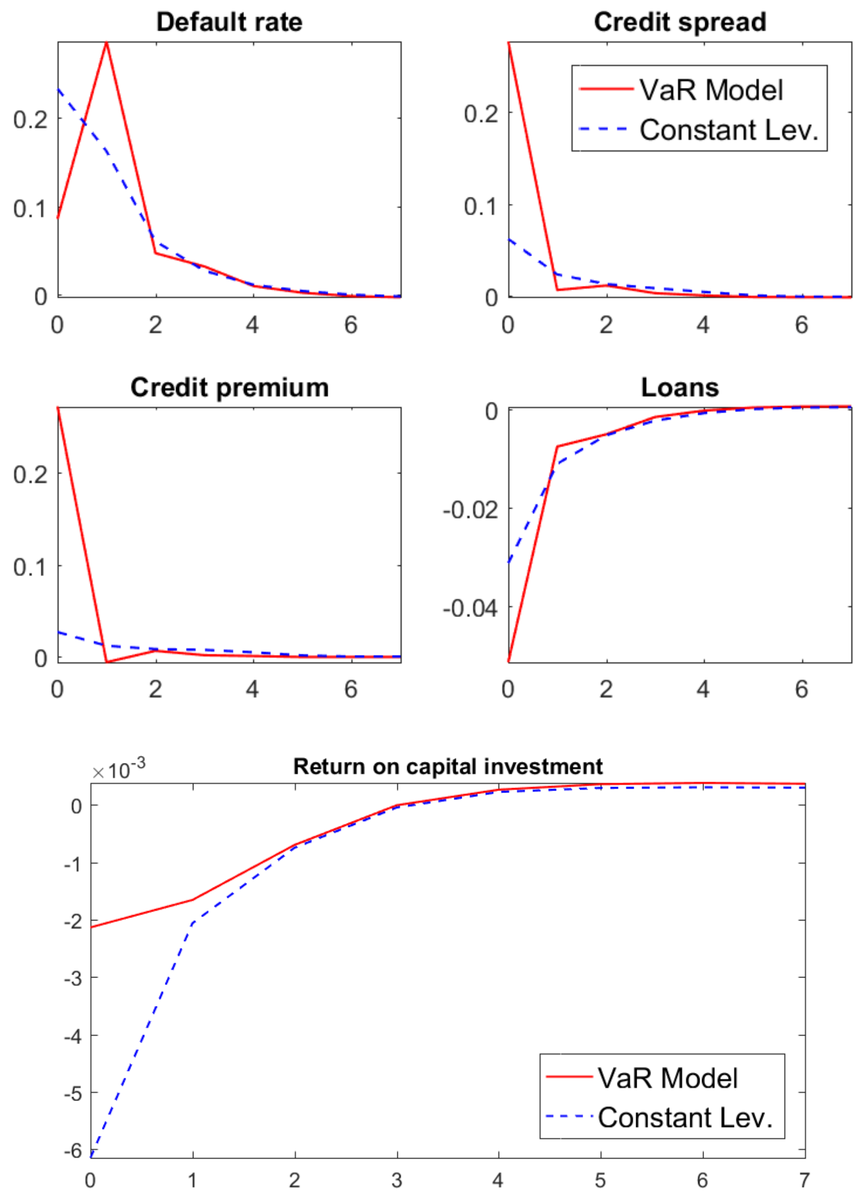

FIGURE 4. IMPULSE RESPONSE FUNCTIONS FOLLOWING AN UNEXPECTED NEGATIVE SHOCK TO TFP

Note: Impulse responses of default rates, credit spreads, credit premia, loans and the return on capital investment R^I in the main model as calibrated in section IV.A (red, continuous line) and a version of the model assuming constant leverage ($\lambda_t^B = \lambda^B$; blue, dashed lines) following $-1\times$ standard deviation unexpected shock to log TFP. All variables are in logarithmic form and are expressed as deviations from the steady state.

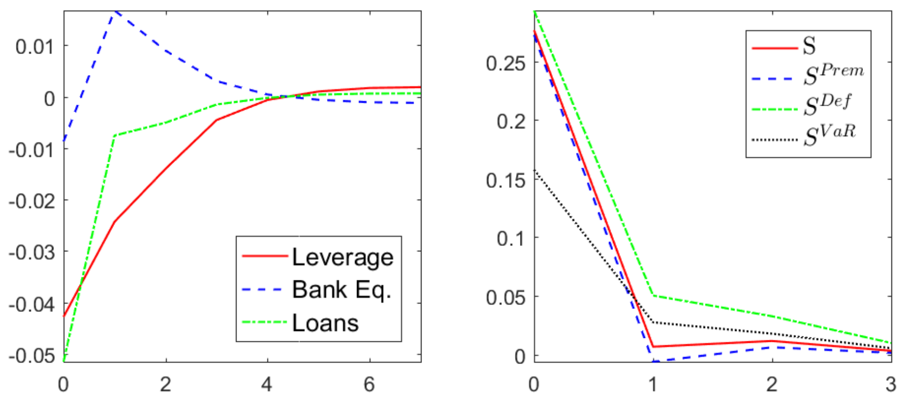

FIGURE 5. THE IMPULSE RESPONSE OF LEVERAGE, EQUITY AND BANK LENDING FOLLOWING A NEGATIVE SHOCK TO TFP

Note: Impulse responses of leverage λ^B, bank equity E^B and lending X in the main model as calibrated in section IV.A following $-1\times$ standard deviation unexpected shock to log TFP. All variables are in logarithmic form and are expressed as deviations from the steady state.

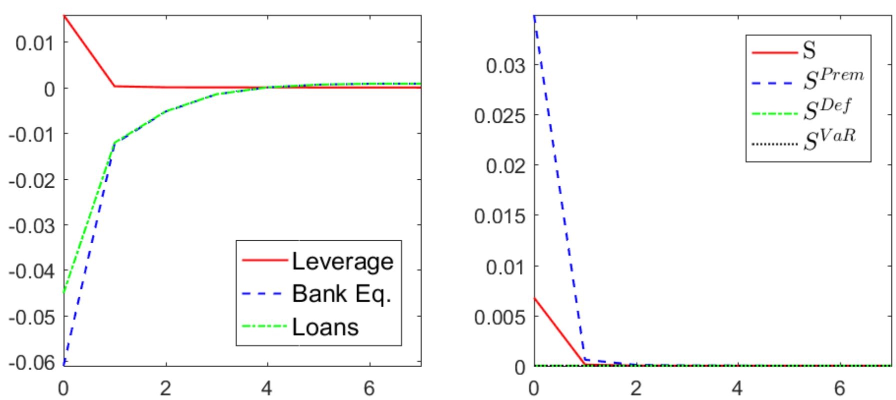

FIGURE 6. THE IMPULSE RESPONSE FUNCTIONS UNDER A HIGH STEADY-STATE DEFAULT ASSUMPTION.

Note: Impulse responses of leverage λ^B, bank equity E^B, lending X, the credit spread S, credit risk premia S^{Prem}, fundamental credit spreads S^{Def} and the VaR-scenario credit spreads S^{VaR} in a model that assumes no adjustment cost in capital production $\kappa_K = 0$ and a riskier capital production process $\sigma_K = 10\%$ following $-1\times$ standard deviation unexpected shock to log TFP. The remaining model's parameters are the same as those as calibrated in section IV.A. All variables are in logarithmic form and are expressed as deviations from the steady state.

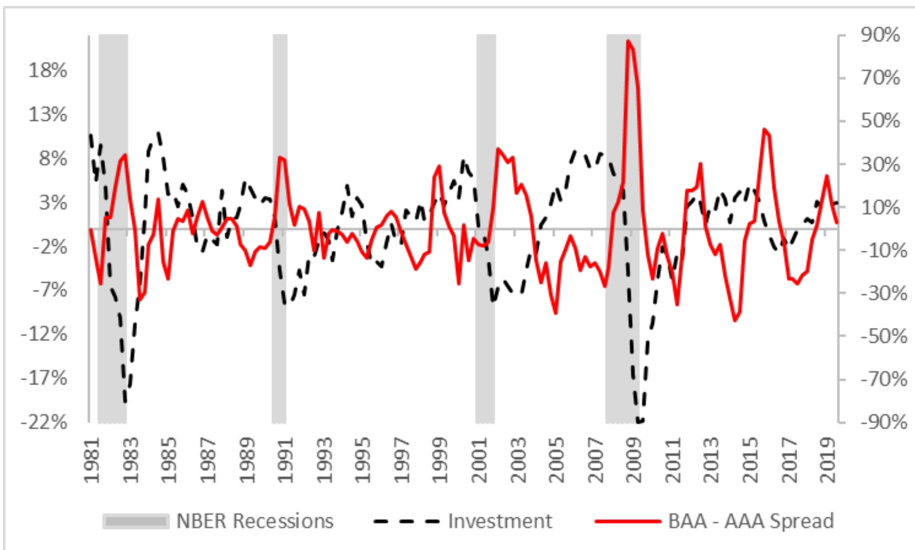

Figure B1. BAA-AAA spreads and aggregate investment in the United States.

Note: Data from Q1 1981 to Q2 2019. Both variables are expressed in log form and as deviations from the trend. Shaded areas represent NBER recessions.
Source: U.S. Bureau of Economic Analysis and Moody's.

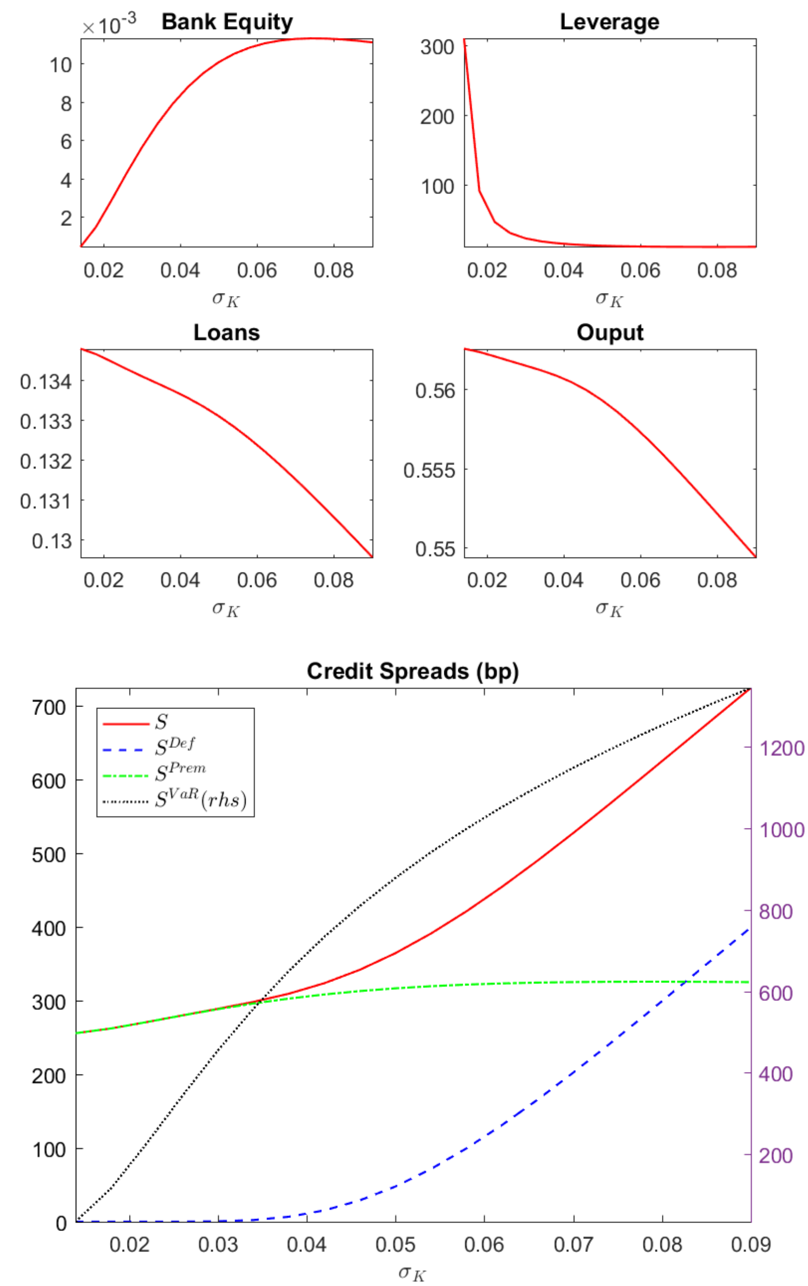

FIGURE D1. THE EFFECTS OF RISKINESS OF CAPITAL PRODUCTION (σ_K) ON STEADY-STATE VARIABLES.

Note: The effect of the riskiness of capital production σ_K on the steady-state values of bank equity E^B, bank leverage λ^B, loan financing X, output Y, the credit spread S, the default risk credit spread S^{Def}, the credit risk premium S^{Prem} and the VaR scenario credit spread S^{VaR} (rhs axis). The remaining model parameters are the same as in table 2.

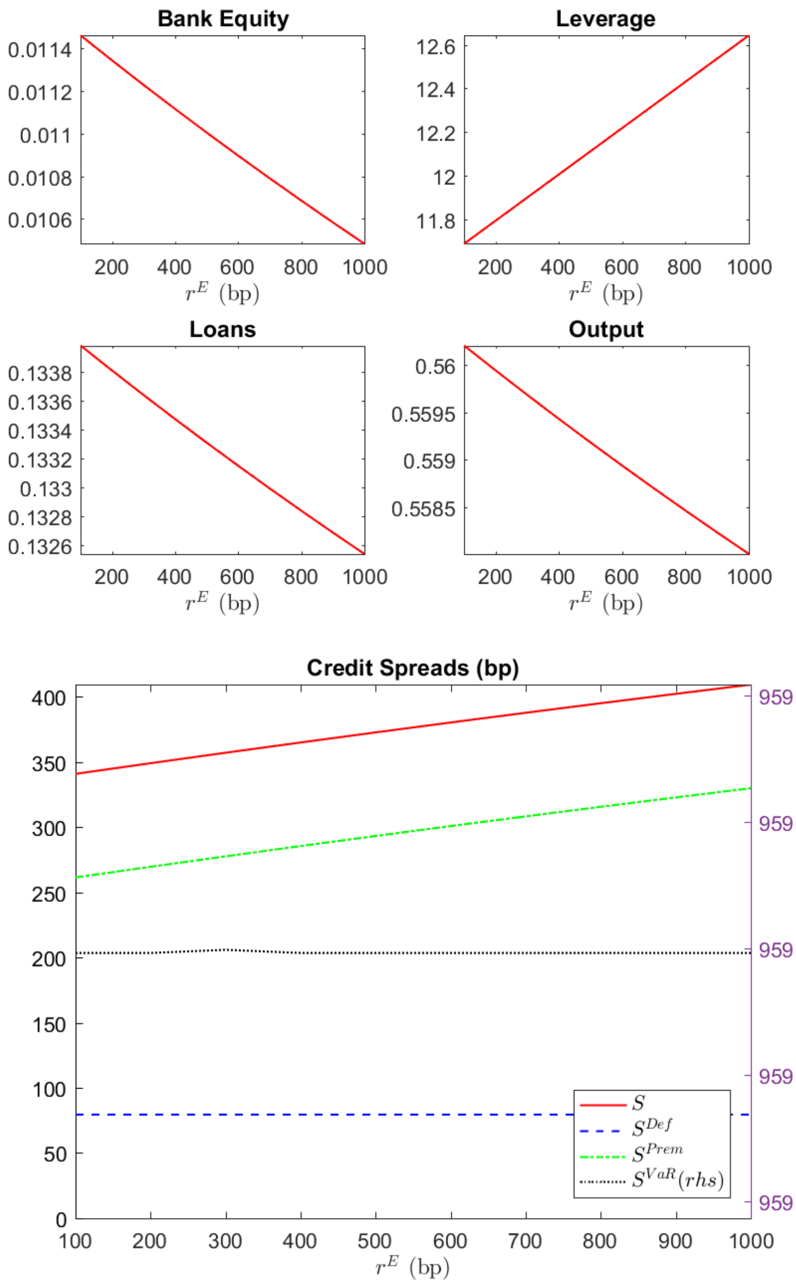

FIGURE D2. THE EFFECTS OF BANKS' EQUITY RISK PREMIUM (r^E) ON STEADY-STATE VARIABLES.

Note: The effect of the bank's equity risk premium r^E on the steady-state values of bank equity E^B, bank leverage λ^B, loan financing X, output Y, the credit spread S, the default risk credit spread S^{Def}, the credit risk premium S^{Prem} and the VaR scenario credit spread S^{VaR} (rhs axis). The remaining model parameters are the same as in table 2.

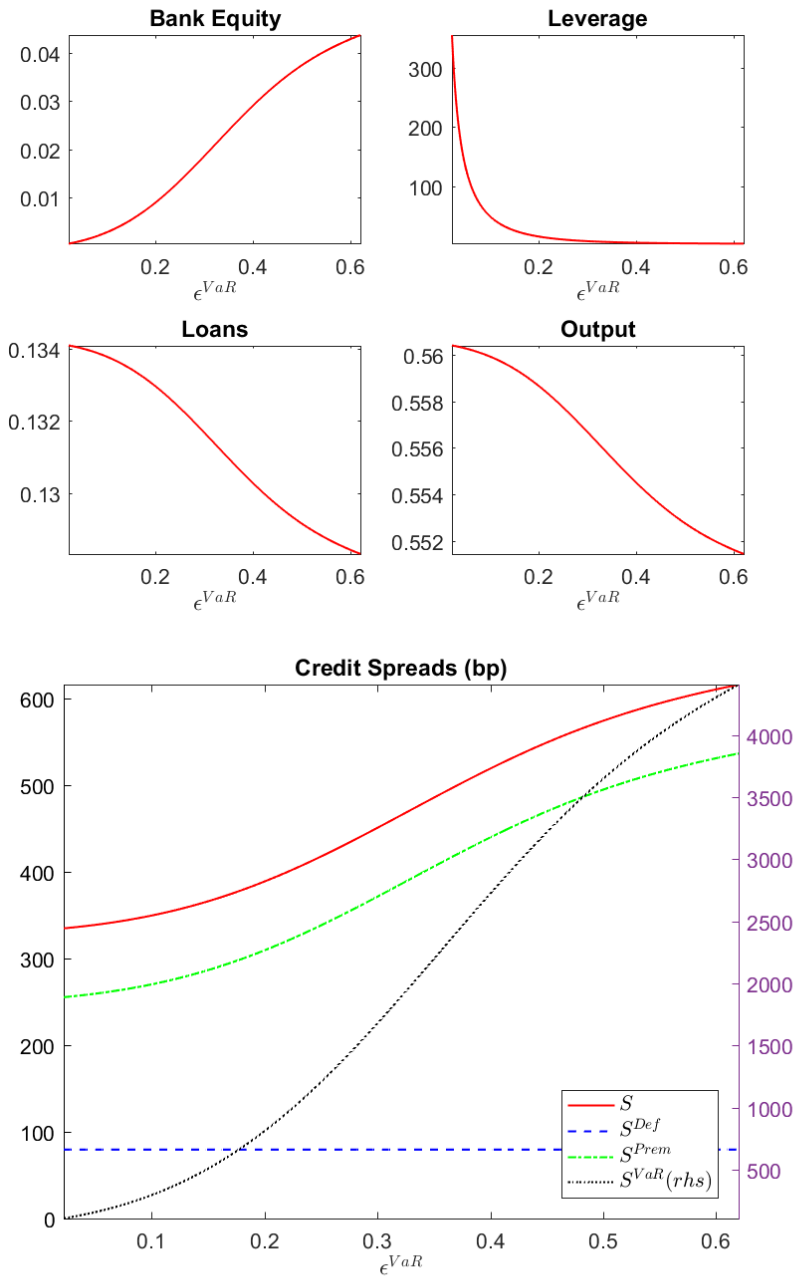

FIGURE D3. THE EFFECTS OF RISKINESS OF THE VAR SCENARIO (ϵ^{VaR}) ON STEADY-STATE VARIABLES.

Note: The effect of the VaR scenario's assumed TFP shock ϵ^{VaR} on the steady-state values of bank equity E^B, bank leverage λ^B, loan financing X, output Y, the credit spread S, the default risk credit spread S^{Def}, the credit risk premium S^{Prem} and the VaR scenario credit spread S^{VaR} (rhs axis). The remaining model parameters are the same as in table 2.

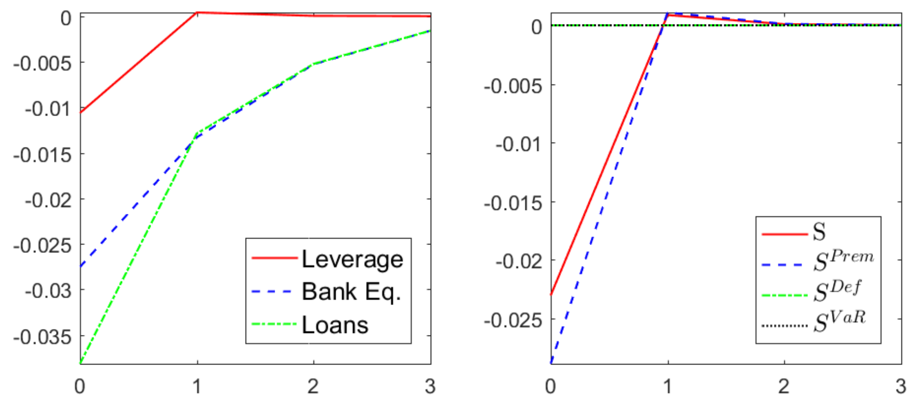

Figure D4. The impulse response functions under no adjustment costs ($\kappa_K = 0$).

Note: Impulse responses of leverage λ^B, bank equity E^B, lending X, the credit spread S, credit risk premia S^{Prem}, fundamental credit spreads S^{Def} and the VaR-scenario credit spreads S^{VaR} in a model that assumes no adjustment cost in capital production $\kappa_K = 0$ following $-1\times$ standard deviation unexpected shock to log TFP. The remaining model's parameters are the same as those as calibrated in section IV.A. All variables are in logarithmic form and are expressed as deviations from the steady state.